OPTIMIZATION TECHNIQUES IN NON-CIRCULAR PROBABILISTIC SLOPE STABILITY ANALYSIS CONSIDERING SPATIAL VARIABILITY

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ABSTRACT
Probabilistic analysis of slopes with several random variables, complex geometries, and spatial variability of soil properties can be carried out using a non-circular random limit equilibrium method (RLEM) search together with an optimization technique. Two optimization techniques are coupled with non-circular slope stability analysis in this paper: 1) the wide-spread Monte Carlo optimization (MCO) technique, and 2) a new surface altering optimization (SAO) method. Both approaches are used to minimize the factor of safety for a given slip surface by modifying its geometry. The SAO approach is a local optimization method which offers flexibility to find the geometry of a critical slip surface using spline curves. In this paper, the results of probabilistic analysis of a tailings dam model using no optimization, optimization using MCO, and optimization using SAO are compared. Results show that the SAO approach can capture more failure paths resulting in larger computed probabilities of failure, and is five times faster than the MCO approach.

RESUME
L'analyse probabiliste des pentes tenant compte de plusieurs variables aléatoires, de géométries complexes et de la variabilité spatiale des propriétés du sol, peut être réalisée en utilisant une méthode de recherche basée sur une méthode d'équilibre limite aléatoire (MELA) non circulaire couplée à une technique d'optimisation. Deux techniques d'optimisation sont couplées à une analyse de stabilité de pente non circulaire dans cet article: 1) la technique d'optimisation de Monte Carlo (OMC) largement répandue, et 2) une nouvelle méthode d'optimisation de modification de surface (OMS). Les deux approches sont utilisées pour minimiser le facteur de sécurité pour une surface de glissement donnée en modifiant sa géométrie. L'approche OMS est une méthode d'optimisation locale qui offre de la flexibilité pour trouver la géométrie d'une surface de glissement critique en utilisant des courbes splines. Dans cet article, nous comparons les résultats de l'analyse probabiliste d'un modèle de barrage de résidus sans optimisation, avec optimisation à l'aide de OMC et avec optimisation à l'aide de OMS. Les résultats montrent que l'approche OMS peut capturer plus de modes de rupture entraînant des probabilités de rupture calculées plus grandes, et est cinq fois plus rapide que l'approche OMC.

1 INTRODUCTION

1.1 Surface Altering Optimization

The search for a critical slip surface is a global optimization problem in which the objective is to minimize factor of safety (F_S) by changing the geometry of trial slip surfaces. Various search algorithms have been proposed in 2D slope stability applications. Some methods utilize brute force, such as grid searching, whereby a discretized region of points is used to define the origins of circular slip surfaces together with different circle radii. This is a lengthy process that checks all areas of the solution space equally, even those regions where global minimums are not present. The method only applies to circular slip surfaces even though slope failures are seldom circular.

The focus in recent years has been on metaheuristic searching (Taha 2010). Many different metaheuristic algorithms have been proposed and tested for 2D slope stability problems (e.g., Gandomi et al. 2015, 2017; Cheng et al. 2007). These metaheuristic algorithms take an input set of slip surface parameters that are then adjusted to minimize the factor of safety using stochastic processes inspired by nature (Gandomi et al. 2016).

Surface altering optimization is a new technique to minimize the factor of safety for a given slip surface by modifying its geometry. This method, similar to the Monte Carlo random walk optimization method (Greco 1996), is a local optimization method. Although SAO can be used independently to find the critical slip surface in slope stability analysis, it works best when combined with a global search method to calculate factor of safety. In this way, the burden of finding an approximate geometry and location of the failure surface with minimum factor of safety is on the global search method; the SAO technique simply modifies the geometry of that surface to further minimize the factor of safety.

To provide flexibility to find the adjusted geometry of more critical slip surfaces, SAO employs spline curves in the 2D analyses. In general, spline formulations require a larger number of parameters in comparison to the number of parameters required to describe primitive geometries such as circles. Therefore, in the larger scheme of slope stability analysis, global optimization methods can be performed using primitive surfaces to find an approximate critical surface relatively fast, followed by SAO using spline functions.

The advantage of SAO over other optimization techniques such as MCO is the accuracy of the results and the reduced computation time. Such performance measures can be investigated using either deterministic and probabilistic slope stability analysis. To best investigate the influence of optimization technique on simulation times, a computationally intensive problem such as a probabilistic analysis considering spatial variability of soil properties is a good choice.

1.2 Spatial Variability

Probabilistic stability analysis results considering spatial variability of soil properties and using limit equilibrium method (LEM) have been reported in studies by El-Ramly et al. (2001), Babu and Mukesh (2004), Cho (2007, 2010), Hong and Roh (2008), Ji et al. (2012), Tabarroki et al. (2013), Li et al. (2014), Javankhoshdel and Bathurst (2014) and Javankhoshdel et al. (2017).

Javankhoshdel et al. (2017) used a circular slip limit equilibrium method and random field theory called the Random Limit Equilibrium Method (RLEM) to investigate the influence of spatial variability of soil properties on probability of failure. Tabarroki et al. (2013) used a non-circular limit equilibrium approach (Spencer method) together with random field theory (non-circular RLEM) to consider spatial variability in their probabilistic analyses.

Cho (2007) carried out probabilistic analysis for a layered slope with spatial variability in soil properties using MCO. He carried out probabilistic analyses with spatial variability of soil properties using the non-circular random limit equilibrium method (RLEM) together with optimization techniques.

There are few studies that investigate the influence of spatial variability of soil properties for the case of layered slopes. Huang et al. (2010) and Cho (2007) investigated the influence of spatial variability on probability of failure using the random finite element method (RFEM) and the non-circular RLEM, respectively. However, the example slopes presented in those studies are slopes with simple geometries.

In this paper, the results of probabilistic analyses of a tailings dam using the RLEM with 1) no optimization, 2) optimization using MCO, and 3) optimization using SAO are computed. Factor of safety, probability of failure, and simulation times are compared for the different cases.

2 SURFACE ALTERING OPTIMIZATION

2.1 Geometry Presentation

In order to minimize the factor of safety for a given slip surface, a proper geometrical formulation to present the slip surface geometry using a finite set of parameters is required. For this purpose, non-uniform rational B-spline (NURBS) can be used. A NURBS curve is defined as a set of weighted control points and a knot vector. A knot vector is a sequence of non-decreasing values dividing the parametric space. The general equation for a NURBS curve is presented as follows (Piegl and Tiller 2012):

$$C(u) = \frac{\sum_{i=0}^{n} N_{i,P}(u) w_i P_i}{\sum_{i=0}^{n} N_{i,P}(u) w_i} \qquad 0 \le u \le 1 \qquad [1]$$

where u is the local parameter that the curve is evaluated at, P_i is a control point in 2D space, n is the number of control points, w_i is the weight of a control point, and $N_{i,p}(u)$ are the p^{th}-degree B-spline functions defined recursively in a knot span $u_i \le u \le u_{i+1}$. (Piegl and Tiller 2012):

$$N_{i,0}(u) = \begin{cases} 1 & u_i \le u \le u_{i+1} \\ 0 & \text{otherwise} \end{cases} \qquad [2]$$

$$N_{i,P}(u) = \frac{u - u_i}{u_{i+P} - u_i} N_{i,P-1}(u) + \frac{u_{i+P+1} - u}{u_{i+P+1} - u_{i+1}} N_{i+1,P-1}(u) \qquad [3]$$

The simplest form of NURBS are first-degree functions that provide a simple presentation of a set of control points connected with straight lines. Increasing the degree of the B-spline yields a smoother curve.

2.2 Structure of SAO

Surface altering is based on a sequence of transformations applied to the geometry of the entire input surface. Each SAO step is solved using the bound optimization by quadratic approximation (BOBYQA) developed by Powell (2009). BOBYQA is a constrained derivative-free optimization method based on trust-region (Wright and Nocedal 2006).

In two-dimensional analysis, a non-circular surface in its simplest form can be described as a linear spline curve. Coordinate values of control points will form the optimization input. As an example, Figure 1 illustrates a surface with 7 control points, yielding 14 input variables to define x and y coordinates of the 2D surface. The geometry of the surface can be altered by modifying these coordinates. SAO offers a systematic set of steps to perform this alteration to minimize the factor of safety while satisfying geometrical convexity and a non-overlapping sequence of control points. These steps are repeated in multiple iterations until convergence criteria are met. A key consideration in SAO is to consider the entire surface geometry, such that changing coordinates of one control point influences the adjustment of the other points to keep the surface convex and to preserve their original order.

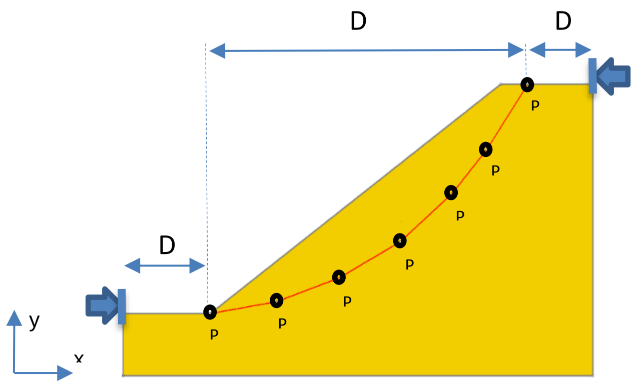


Figure 1. A linear spline 2D slip surface.

2.3 Weak Layers

In the presence of geological structures with weak layers, it is likely that the critical slip surface passes through the weak layer. Thin structured weak layers introduce an extra challenge to find a set of proper transformations applied to the slip surface such that it extends through the weak layer. In the presence of weak layers, an additional step can be performed. Whenever the y-coordinate of the control point is altered, an additional displacement can be applied such that the control point moves inside the weak layer. If this additional displacement results in a smaller factor of safety, then the change is accepted.

2.4 Comparison with Monte Carlo Optimization

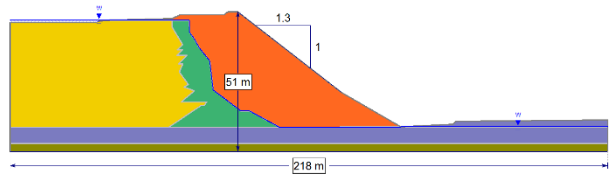
Surface optimization based on Monte Carlo random walk (Greco 1996) relies on modifying the positions of slip surface control points in two phases identified as exploration and extrapolation. In the exploration phase, a new location for each control point is created randomly by slightly replacing each control point. If the new slip surface results in a lower factor of safety, the change is preserved, otherwise it returns the control point to its previous position. In the extrapolation phase it applies computed displacements for all the control points after exploration and keeps the change if it yields a lower factor of safety. Iterations stop when changes in the computed factor of safety fall below a given prescribed threshold. Although random generation in MCO is somewhat improved by adding the extrapolation phase, in general it requires a large number of factor of safety evaluations until an optimal solution is reached, typically 5 to 10 times more than those required using SAO.

3 EXAMPLE: SPATIAL VARIABILITY

3.1 The Tailings Dam Model

The Mount Polley tailings dam was selected to provide baseline geometry and soil properties for this study (Province of British Columbia, 2015). It is important to note that the purpose of this example is not to re-analyze the failure of the dam, but to use the dam as a baseline case to compare the output results using different optimization cases. Some soil properties were adjusted from the Mount Polley case study so that detectable values of probability of failure could be computed for the purpose of comparison of deterministic and probabilistic analysis outcomes.

The model used in this study is taken from Cami et al. (2017), using data provided by Province of British Columbia (2015). The model and baseline material parameters used in the current study are shown in Figure 2.



Material Name	Color	Unit Weight (kN/m3)	Strength Type	Cohesion (kPa)	Phi (deg)	Water Surface	Hu Type	Hu	Ru
Tailings		18	Mohr-Coulomb	0	30	Water Surface	Custom	1	
Core		20.5	Mohr-Coulomb	0	35	Water Surface	Custom	1	
Rock		22	Mohr-Coulomb	100000	0	None			0
Upper Till		21	Mohr-Coulomb	0	35	Water Surface	Custom	1	
Upper Glaciolacustrine		20	Mohr-Coulomb	25	20	Water Surface	Custom	1	

Figure 2. The Mount Polley tailings dam model and the material properties used in this study.

3.2 Spatial Correlation Length

Liu and Chen (2010), Lloret-Cabot et al. (2014), and Pieczynska-Kozlowska (2015) measured spatial correlation lengths using field CPT data. They showed that due to the larger number of measurements available in the vertical direction, the calculated value of vertical correlation length can be found relatively accurately. The horizontal correlation length on the other hand is harder to determine, particularly when measurement locations are far apart.

In this study, vertical correlation length was calculated using a method outlined by Vanmarcke (1977) which estimates the correlation length by minimizing the error between the theoretical and empirical correlation models.

The empirical correlation model, $\hat{\rho}(\tau)$, is shown in Equation 4:

$$\hat{\rho}(\tau_k) = \frac{1}{\hat{\sigma}^2 (n-k)} \sum_{t=1}^{n-k} (x_t - \hat{\mu})(x_{t+k} - \hat{\mu}) \qquad [4]$$

where k is the number of lag distances (τ) between two points, $\hat{\sigma}$ is the estimated standard deviation of the detrended CPT data, n is the number of observations, x_t is the detrended tip resistance value at location t, and $\hat{\mu}$ is the estimated mean of the detrended CPT data.

In this study the Markov correlation model (Equation 5) was used to calculate the theoretical correlation values.

$$\rho(\tau_k) = \exp\{\frac{-2|\tau_k|}{\theta}\} \qquad [5]$$

where θ is the correlation length.

Using the method described above, the vertical correlation length was measured at nine different CPT locations (Province of British Columbia, 2015) and found to range between 0.3 m and 1.8 m, with most values falling closer to 1 m; hence, a vertical correlation length of 1 m was used in this study. A square mesh size of 0.5 m was used in the random field to accommodate this correlation length. An example field from one set of tip resistance data is shown in Figure 3. The corresponding correlation length is 0.86 m.

Due to less data in the horizontal direction as noted earlier, the effect of horizontal correlation length was assumed to be infinity in this study.

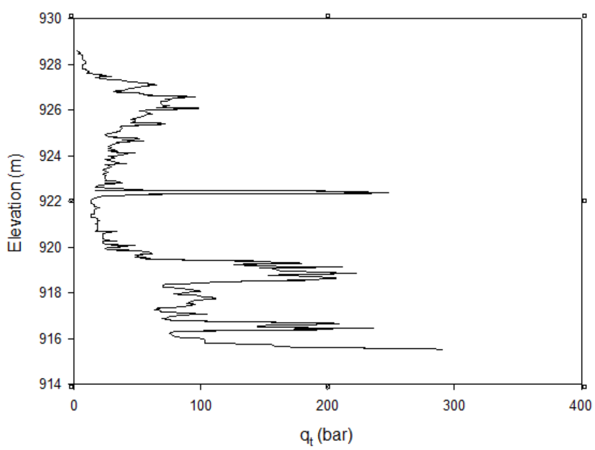


Figure 3. Tip resistance data. The vertical correlation length calculated for this case is 0.86 m.

3.3 Slope Stability Analysis using LEM

In this study, the Morgenstern-Price limit equilibrium method was used with the half sine interslice force function to calculate factor of safety. 50 slices for the probabilistic analysis were used in this study. 10,000 simulations were used for all probabilistic analyses. A deterministic analysis case (section 3.3.1) and a RLEM case (section 3.3.2) are examined.

The non-circular auto refine search method was used to locate the initial minimum slip surface before optimization (Rocscience Inc., 2015). With this method, the search for the lowest safety factor is refined as the search progresses such that the results of one iteration are used to narrow the search area on the slope in the next iteration.

The auto refine search method for non-circular surfaces first generates circular surfaces using the algorithm described below:

1. The slope surface is divided into a number of divisions.
2. Circles are generated between each pair of divisions, according to the required level of accuracy (generally ten circles are enough) (Figure 4).
3. The safety factors are then calculated for these circles using a limit equilibrium approach such as the Morgenstern-Price method, and the average safety factor associated with each division along the slope, is recorded.
4. This constitutes one iteration of the auto refine search method.
5. Half of the divisions along the slope are used to define a new, narrowed search area, for the next iteration. Only the divisions with the lowest (average) safety factors are used, while the divisions with high safety factors are discarded from the analysis.
6. The divisions of the slope which are retained, are then used to form a new slope polyline. Using this new, narrowed slope surface, steps 1 to 5 are then repeated, for 10 more iterations.

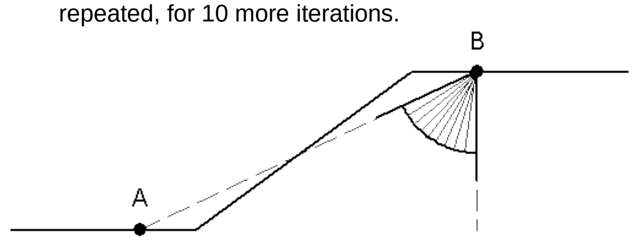


Figure 4. Division circles in the auto refine search.

The circular surfaces have now been generated. Each circle is converted into a non-circular (piece-wise linear) surface with a number of vertices (10 vertices are used in this study) and the F_s is calculated for each non-circular surface. The slip surface with the lowest F_s is determined using the circular algorithm outlined above. An optimization search can be applied at this stage.

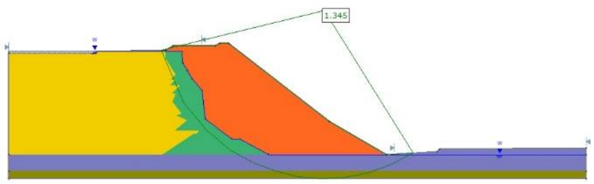
To investigate the influence of each optimization technique on probability of failure and also compare the simulation time for each technique, the combination of non-circular auto refine search with three different optimization cases were considered in this study: 1) no optimization 2) optimization using MCO, and 3) optimization using SAO. Cases 2 and 3 are especially effective at locating (searching out) slip surfaces with lower safety factors, when used in conjunction with a non-circular search. These three cases are applied to a deterministic analysis and a RLEM analysis. The program *Slide* 2018 (Rocscience 2018) was used to carry out the computations.

3.3.1 Deterministic Analysis

In the deterministic slope stability analysis example, the material parameters shown in Figure 2 are assumed to be constant values that do not vary within each material unit.

The results of deterministic analyses using the three different cases of no optimization, optimization using MCO and optimization using SAO are presented in Figure 5. The factor of safety for the case with no optimization was 1.35 while the factor of safety for the cases with MCO and SAO techniques was 1.26 and 1.26, respectively.

a)



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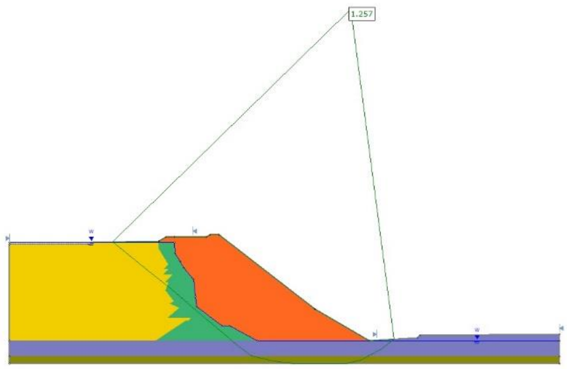


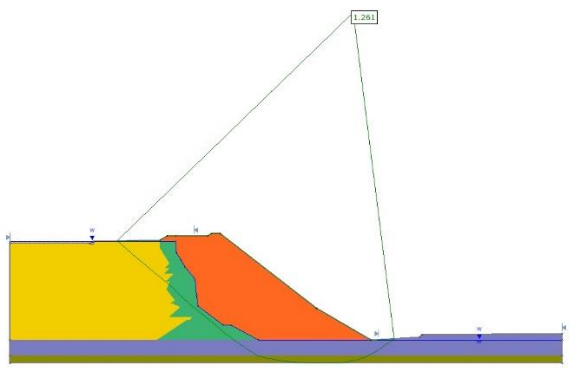


Figure 5. Critical slip surface geometry from deterministic analyses using a) no optimization, b) MCO, and c) SAO.

A deterministic shear strength reduction (SSR) analysis was also computed in order to verify the LEM results. Figure 6 shows the maximum computed shear strain contours which correspond to the model, which resulted in a shear strength factor (SRF) of 1.26 which is in a good agreement with the results of LEM with optimization techniques. The surface shown in Figure 6 is also in agreement with the critical non-circular LEM slip surface.

3.3.2 RLEM

In the RLEM, a random field is first generated using the local average subdivision (LAS) method developed by Fenton and Vanmarcke (1990) and then mapped onto a grid of elements (mesh). Each mesh element in the random field has different values of soil properties, and cells close to one another have values that are closer in magnitude, based on the value of the spatial correlation length. In each realization, a search is carried out to find the mesh elements intersected by the slip surface. The random soil property values are assigned to the slices whose base mid-point falls within that element. A limit equilibrium approach (Morgenstern-Price method) is then used to calculate the factor of safety for each realization. The probability of failure (Pf) is calculated as the ratio of the number of simulations resulting in Fs < 1 to the total number of simulations.

A combination of the COV of soil properties was selected (Table 1), i.e. COV_γ was set to the typical maximum value of 0.1, COV_ϕ was set to the typical maximum value of 0.2 and COV_c was set to a typical value of 0.3. Table 2 shows the random variable parameters used in this study. Lognormal distributions are assumed for all random variables. The core and rock materials were assumed to have the constant material properties defined in Figure 2.

Table 1. COV values used in probabilistic analyses.

Parameter	COV_γ	COV_ϕ	COV_c
Value	0.1	0.2	0.3

Table 3 summarizes the probability of failure results from non-circular RLEM analyses with different optimization cases. In these analyses, the soil properties were expressed as anisotropic random fields with spatial variability in the vertical direction only. The vertical correlation length was taken as 1 m as noted earlier. Figure 7 shows the failure surface band for the three different optimization cases. The deterministic failure surface presented in Figure 5 is also shown in this figure.

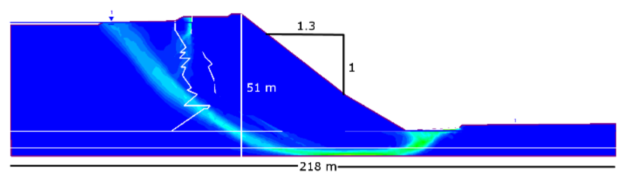
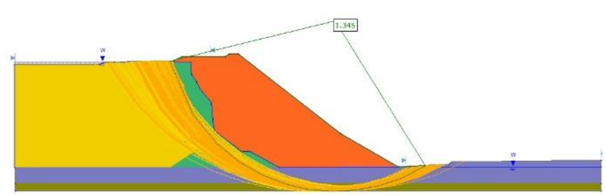


Figure 6. Maximum shear strain contours from an example SSR analysis. SRF = 1.26.

a)



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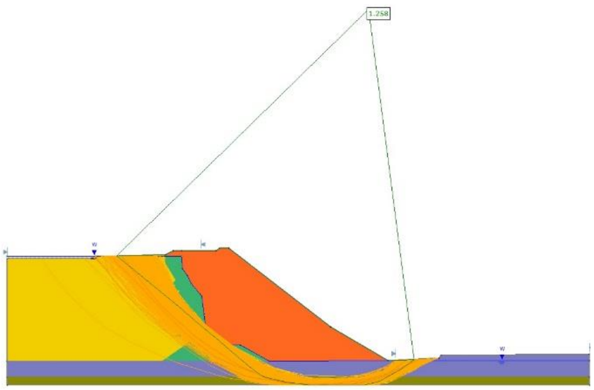

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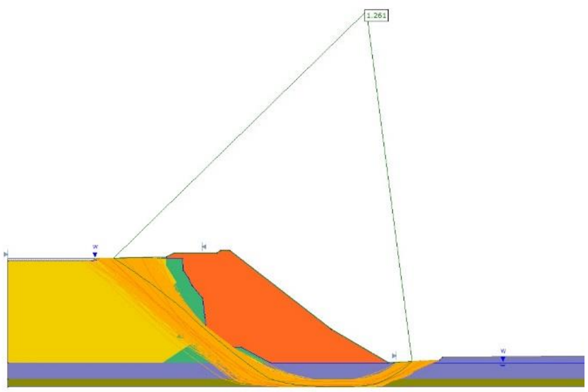


Figure 7. Results of probabilistic analyses: a) no optimization, b) MCO technique, and c) SAO technique.

It can be seen in Table 3 that, for the no optimization case, P_f is zero. However, the combination of auto refine search method and optimization helps to find weaker failure paths. P_f for the analyses with MCO is 0.25%, compared to the case with SAO in which P_f = 0.47%. It can be observed from these results that SAO improves the auto refine method, i.e. finds more failure surfaces compared to the case without optimization and also the case with MCO.

Table 2. Random variable parameters. Lognormal distributions are used with all variables.

Material	Property	Mean	Std. Dev.
Tailings	Unit weight (kN/m^3)	18	1.8
Tailings	Friction angle (degrees)	30	6.0
Upper Till	Unit Weight (kN/m^3)	21	2.1
Upper Till	Friction angle (degrees)	35	7
Upper Glaciolacustrine	Unit weight (kN/m^3)	20	2.0
Upper Glaciolacustrine	Friction angle (degrees)	20	4.0
Upper Glaciolacustrine	Cohesion (kPa)	25	7.5

Also, assuming that a reasonable target design value for probability of failure for slopes is P_f = 0.01% (Silva et al. 2008), using no optimization implies that the slope is safe. However, MCO and SAO techniques show that the failure probability is greater than the target design value and thus the slope is not safe for design. It can also be noted that the P_f calculated using SAO is about three times greater than that calculated using MCO indicating that results calculated using SAO are upper bound values for design.
Table 3 also shows the computation time for each method. The main advantage of SAO compared to MCO is the great improvement in speed. It can be seen in Table 3 that SAO reduces simulation times by about a factor of 5 compared to MCO. This is an important benefit for probabilistic analysis. Thus, SAO not only improves the results by finding more failure surfaces, but also reduces the computational expense.

Figure 8 shows the failure surfaces corresponding to the lowest factor of safety in the probabilistic analysis using the three optimization cases. It is shown that the SAO technique found a minimum factor of safety less than 1 (F_s = 0.972), while the MCO method was not able to find the same surface (F_s = 1.06 and 1.16, with MCO and no optimization, respectively). It can be seen in the figure that both SAO and MCO failure mechanisms pass through the weak layer, but analyses with SAO were still able to find mechanisms with lower factor of safety compared to MCO.

4 CONCLUSION

This study provides an introduction to the surface altering optimization (SAO) technique in slope stability analysis to calculate factor of safety. This optimization technique is used together with a general (limit equilibrium) approach to calculate factor of safety such that the burden of finding an approximate geometry and location of the failure surface with minimum factor of safety falls on the global search method. The SAO technique is employed to modify the geometry of that surface in order to further minimize the factor of safety.

To demonstrate the accuracy of SAO, F_s values were first compared to results using the more common MCO technique as well as the shear strength reduction method. Good agreement was found between the three results.

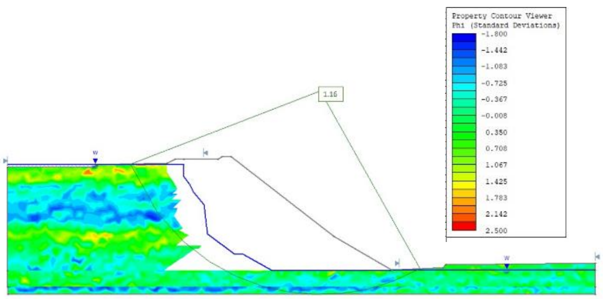
A computationally intensive probabilistic analysis was carried out on a tailings model with different layers of soil and considering spatial variability of soil properties. It was shown that the SAO method gave upper bound values of probability of failure (i.e., found more critical failure surfaces) compared to the case with MCO and the case with no optimization. The simulation time was improved by a factor of 5 using SAO compared to MCO.

Finally, the no optimization case implied that the slope was safe for design (P_f = 0% ≤ 0.01%). However, combining the limit equilibrium method with the optimization techniques described in this paper showed that the probability of failure was greater than the minimum target P_f assumed for design.

Table 3. RLEM analyses results.

Parameter\Opt.	RLEM no opt.	RLEM MCO	RLEM SAO
Det. F_s	1.35	1.26	1.26
Mean F_s	1.31	1.2	1.19
P_f (%)	0	0.25	0.47
Simulation time (hours)	1.5	22	4.5

a)



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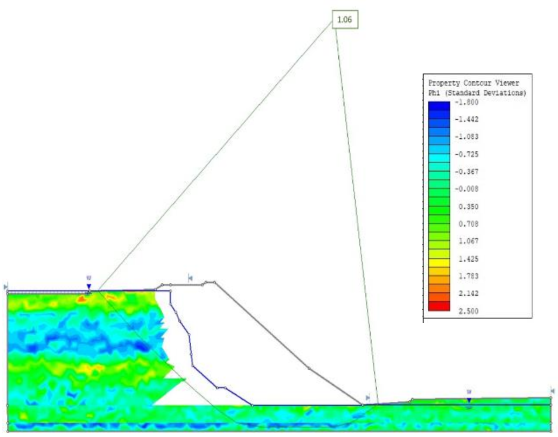

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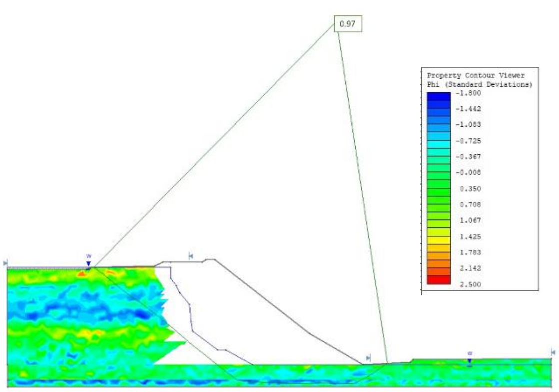

Figure 8. Spatial field and a failure mechanism with the lowest F_s using SAO and the same field for other methods: a) no optimization, b) MCO technique, and c) SAO technique.

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