ON STATISTICAL INFERENCE OF THE COEFFICIENT OF CONSOLIDATION BY USE OF NUMERICAL OPTIMISATION

Bhamisha Ramdharry, Faculté d'ingéniérie, Université de Moncton, Moncton E1A 3E9
Paul Chiasson, Faculté d'Ingéniérie, Université de Moncton, Moncton, NB, E1A 3E9 chssp@umoncton.ca

ABSTRACT

A Least Squares method is developed to estimate the coefficient of consolidation from oedometetric data. It involves the comparison of the theoretical and experimental values of the average degree of consolidation based on the Terzaghi one-dimensional theory of consolidation. The sum of the squares of residuals between the two factors are minimised using a Best Linear Unbiased Estimator (BLUE) under the condition that the sum of the residuals is zero. Experimental data was collected to calculate the coefficient of consolidation by existing methods and the Least Squares method. The results are found to be in good agreement, proving that the proposed method can be an alternative way of determining the coefficient of consolidation.

RÉSUMÉ

Une méthode par moindres carrés est développée afin d'estimer le coefficient de consolidation des mesures d'un essai à l'oedomètre. Elle consiste à comparer les valeurs théoriques et expérimentales du pourcentage de consolidation moyen qui sont dérivées de la théorie unidimensionnelle de Terzaghi. Le principe repose sur l'application d'un estimateur optimal non biaisé. Le coefficient de consolidation estimé minimise la somme des carrées de la différence entre les valeurs théoriques et expérimentales, tout en satisfaisant la contrainte que la somme de ces différences soit nulle. Des données expérimentales ont été recueillies pour calculer le coefficient de consolidation par les différentes méthodes disponibles, ainsi que par la méthode des moindres carrés. On peut déduire que les résultats de la méthode proposée sont comparables. La méthode est donc une approche alternative à la détermination du coefficient de consolidation.

1. INTRODUCTION

There has been considerable research done in determining alternative methods for computing the coefficient of consolidation from laboratory oedometer tests. Certainly the most commonly used are the logarithm of time fitting method (Casagrande and Fadum 1940) and the square root of time fitting method (Taylor 1948). Other methods found in the literature are the Inflection Point method (Mesri et al. 1999), the Rectangular Hyperbola method (Sridharan et al. 1987), and the coefficient of consolidation from the linear segment of the t½ method (Feng and Lee 2001), which are all discussed further in this paper. In spite of the existence of all these methods, there is still a need for a method which is independent of judgement (Casagrande's method) or of irregularities in the shapes of curves due to secondary compression (Inflection Point method). As such, the Least Squares method has been elaborated to provide optimal unbiased results and structured to be independent of the judgement of the interpreter of the experimental data. This is possible via the use of a Best Linear Unbiased Estimator (BLUE).

2. LEAST SQUARES METHOD

The Least Squares method consists of curve fitting the experimental average degree of consolidation, U_{avg} versus the time factor, T by means of the coefficient of consolidation obtained from the Casagrande or Taylor methods as seed value, and adjusting it with the theoretical Terzaghi T-U_{avg} curve using, as mentioned above, a Best Linear Unbiased Estimator (BLUE) which minimises the sum of the squares of residuals under the condition that the sum of the residuals is zero.

2.1 Existing methods for the determination of C_V.

Casagrande's method is based on the identification of the inflection point, as well as the compression values corresponding to 0% and 100% consolidation, from which the time corresponding to 50% consolidation, t_{50} can be deduced and used for the determination of the coefficient of consolidation.

$$C_V = \frac{0{,}197\, H_{dr}^2}{t_{50}} \qquad [1]$$

Casagrande's method, though widely used, poses two concerns: the use of personal judgement in the identification of the inflection point from which a tangent is drawn and the possibility of secondary compression occurring before t_{100}.

Taylor (1948) proposes that the end of primary compression occurs at t_{90}. This value is obtained by identifying the linear portion approximately up to U = 60% and drawing a line with the abscissa being 1.15 times that of the line previously obtained. The time corresponding to

90% consolidation and deformation d_0 at t = 0 can thus be determined. But in this case also, secondary compression might affect the shape of the curve and the results for the value of the coefficient of consolidation.

The coefficient of consolidation from the linear segment of the t½ curve (Feng and Lee 2001) requires plotting the curve of U_{avg} versus $T^{1/2}$. The initial part of the curve is a straight line up to about 60% consolidation where the curve deviates from the straight line. The coefficient of consolidation can then be calculated using t_{60}.

In the Inflection Point method (Mesri et al. 1999), the compression versus log time curve is plotted and the inflection point is visually identified. This corresponds to an average degree of consolidation of 70%. However, this method is less reliable because of the personal judgement factor and the possibility of having no inflection point due to the presence of secondary compression. Note that the effect of secondary compression is minimised in the last two methods by having their reference point taken at an average degree of consolidation near 50%.

The Rectangular Hyperbola method (Sridharan et al. 1987) implies that a straight line is obtained between the average degrees of consolidation points of 60% and 90% on a time/compression versus time graph. This method does not involve the difficulties encountered with the Taylor or Casagrande method resulting from irregular shapes of curves.

2.2 Proposed interpretation method

The Terzaghi theory of one-dimensional consolidation defines the relationship between the theoretical time factor, T and the average degree of consolidation, U_{avg}. The solution of this relationship for an initial excess pore water pressure constant with depth is

$$U_{avg} = 1 - \sum_{m=0}^{\infty} \frac{2}{M^2} \times \exp(-M^2 \times T) \qquad [2]$$

where

$$M = \frac{\pi(2m+1)}{2} \qquad [3]$$

The coefficient of consolidation can then be calculated from Equation [4] where t is the consolidation time and H_{dr} the drainage distance:

$$T = \frac{C_v \times t}{H_{dr}^2} \qquad [4]$$

With the consolidation time from experimental data, the time factor T is calculated using Equation 4 with the values of the coefficient of consolidation obtained from the Casagrande or the Taylor procedure as initial value. The theoretical values of the average degree of consolidation are calculated as follows:

For U ≤ 60%

$$T = \frac{\pi}{4} \times U^2_{avg} \qquad [5a]$$

Rearranged in function of T, this gives

$$U_{avg} = \sqrt{\frac{4 \times T}{\pi}} \qquad [5b]$$

For U > 60%

$$T = 1.781 - 0.933 \times \log_{10}[(1 - U_{avg})100] \qquad [6a]$$

Rearranged in terms of T, this gives

$$U_{avg} = 1 - 10^{\left(\frac{T-1.781}{0.933} - 2\right)} \qquad [6b]$$

The theoretical curve of U_{avg} versus T is plotted. The experimental values of the degree of consolidation are calculated using Equation 7 or Equation 8 These two equations were explored following methods of Casagrande and Taylor.

$$U_{avg,\,exp} = \frac{d - d_0}{d_{100} - d_0} \times 1 \qquad [7]$$

$$U_{avg,\,exp} = \frac{d - d_0}{d_{90} - d_0} \times 0.9 \qquad [8]$$

where d is the compression value at time t, d_0 is the initial compression, d_{90} and d_{100} are the compression values at average degree of consolidation of 90% and 100% respectively. The experimental curve can now be plotted.

The fitting between the theoretical and experimental curves is achieved by using a Best Linear Unbiased Estimator (BLUE) which is used to evaluate unknown parameters via iterations from the initial values obtained from the Casagrande or Taylor procedure: d_0, d_{100} or d_{90} and C_V. The residual between the estimator of the function $U_{avg}(T)$ and the measured average degree of consolidation $U_{avg,\,exp}$ is found from Equation 9.

$$\varepsilon_j = U_{avg}(T)_j - (U_{avg,\,exp})_j \quad [9]$$

The solution for d_0, d_x (x = 90 or 100) and C_v is obtained by minimising the sum of squared residuals

$$\text{Min} \sum_{j=1}^{n} \varepsilon_j \times w_j \quad [10]$$

with the unbiased condition of

$$\sum_{j=1}^{n} \varepsilon_j \times w_j = 0 \quad [11]$$

where n is the number of measurements and w_j is the weight of the residual. The weight allows control over secondary compression. The analysis of the experimental data has been performed with a weight threshold of 60% and 90%. If, for example it is decided that secondary compression affects consolidation as from 60%, the weight of the residual is set to 1 if the measured average degree of consolidation is below 60% and the deviation of the measured average degree of consolidation from the estimated value is accepted. If the measured average degree of consolidation is above 60%, the weight is set to 0, and the residual is 0 (i.e. rejected). The weight of the residual can also be varied and therefore allows the elimination of suspected secondary compression in the calculation of the coefficient of consolidation.

The complexity of the equation of consolidation (Equation 2) or the approximate Equations 5b, 6b does not allow the development of an analytical solution to this problem. Numerical optimization techniques are therefore required. These are readily available in most modern spreadsheets.

The following summarises the least Squares method

1. Compute initial values for d_0, d_x (x = 90 for Equation 8 or 100 for Equation 7) and C_v from the method of Casagrande or Taylor.
2. Compute $(U_{avg,exp})_j$ from the experimental compression measurements d_j using Equations 7 or 8 accordingly.
3. Compute T_{exp} from experimental times t_j of measurements using Equation 4 and C_v of step 1.
4. Compute U_{avg} for T_{exp} using Equation 5b and [6b].
5. Compute the residuals using Equation 9, sum of squared residuals using Equation 10 and the sum of residuals using Equation 11.
6. Repeat steps 2 to 5 until a minimum for Equation 10 is found that satisfies the constraint of Equation 11. This is done through numerical optimisation.

Table 1. Experimental data used [Data # 1 and 5 Holtz and Kovacs (1991), Data # 2 Robitaille and Tremblay (1997), Data # 3 and # 4 Bowles (1992), Data # 6 Liu and Evett (2003), Data # 7 Wray (1986)

Experimental data													
#1		#2		#3		#4		#5		#6		#7	
Initial height of specimen (H_0)before adding load increment (mm)													
21.870		25.1060		20.004		20.016		25.400		19.8100		27.653	
Time (min)	d (mm)	Time (min)	d (mm)	Time (min)	d (mm)	Time (min)	d (mm)	Time (min)	d (mm)	Time (min)	d (in.)	Time (min)	d (in.)
0	6.627	0	0.1651	0	0.590	0	0.9800	0	4.041	0	0	0	0.827
0.1	6.528	0.25	0.2100	0.1	0.660	0.1	0.1095	0.1	3.927	0.1	0.0067	0.25	0.871
0.25	6.480	0.5	0.2121	0.25	0.675	0.25	0.1120	0.25	3.879	0.25	0.0069	0.5	0.873
0.5	6.421	1	0,2151	0.5	0.695	0.5	0.1135	0.5	3.830	0.5	0.0071	1	0.876
1	6.337	2	0.2184	1	0.700	1	0.1160	1	3.757	1	0.0077	2	0.882
2	6.218	4	0.2223	2	0.720	2	0.1175	2	3.650	2	0.0084	4	0.889
4	6.040	8	0.2273	4	0.730	4	0.1200	4	3.495	4	0.0095	8	0.899
8	5.812	15	0.2311	8	0.750	8	0.1235	8	3.282	8	0.0107	15	0.907
15	5.489	30	0.2350	15	0.780	15	0.1270	15	3.035	15	0.0120	30	0.917
30	5.108	60	0.2388	30	0.800	30	0.1325	30	2.766	30	0.0132	60	0.928
60	4.775	120	0,2426	60	0.835	60	0.1380	60	2,550	60	0.0144	120	0.943
120	4.534	240	0.2464	120	0.885	120	0.1435	120	2.423	120	0.0152	240	0.958
240	4.356	-	-	256	0.940	256	0.1480	240	2.276	240	0.0158	480	0.971
480	4.209	-	-	420	0.970	579	0.1510	505	2.184	480	0.0160	960	0.980
1382	4.041	-	-	1417	0.980	1410	0.1520	1485	2.040	1380	0.0162	1440	0.981

2.3 Tests and results

A typical curve fitting result for the Least Squares method is illustrated in Figure 1. It can be seen that the measured T-U_{avg} curve fits the estimated one perfectly until approximately 90%.

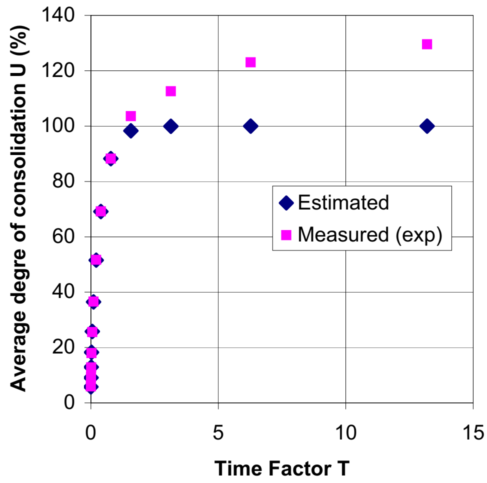

Figure 1. Curve fitting of experimental and estimated average degree of consolidation versus time factor.

Table 1 shows experimental data collected to compare values of coefficient of consolidation obtained by the methods mentioned in this paper.

For each set of data, the coefficient of consolidation has been calculated by each method and compared with the values obtained from the Least Squares method. Some of the results are as shown in Figures 2 and 3.

The values of the coefficient of consolidation are in good agreement with those obtained from the other methods. Those computed from the Least Squares method with d_0, d_{90} and C_V obtained from a seed value of the Taylor method vary from 1 to 1.6 times the values obtained from Taylor's method (Figure 2). When using d_0, d_{100} and C_V obtained a seed value from Casagrande method, the results are between 1.1 and 2.1 times those obtained from Taylor's method (Figure 3).

Comparison of the application of the Least Squares method using the d_0, d_{100} and C_V obtained from a seed value of the Casagrande method and the d_0, d_{90} and C_V from a Taylor method seed is illustrated in Figure 4. In this case, the values differ by a factor of 1.1 to 1.8. The slight difference can be explained by the sensibility of the estimator with respect to initial values d_0, d_{90} or d_{100} and C_V as well as the possible existence of more than one local minimum. The numerical estimator can be trapped in a local minimum. Also, since the initial values are used to calculate $U_{avg,\ exp}$, the difference in the definition between Equation 7 and 8 can in fact define two slightly different problems leading to different solutions. This may be attributable to a more appreciable contribution of secondary compression in the definition of $U_{avg,\ exp}$ in Equation 8 than in Equation 7.

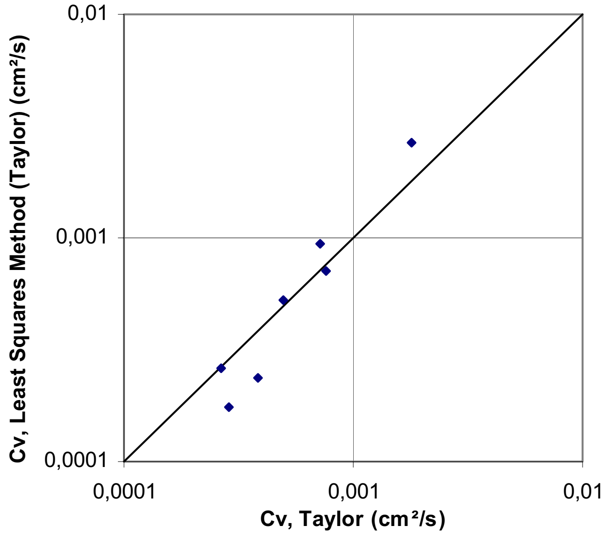

Figure 2. Values of coefficient of consolidation obtained from Least Squares method and Taylor method.

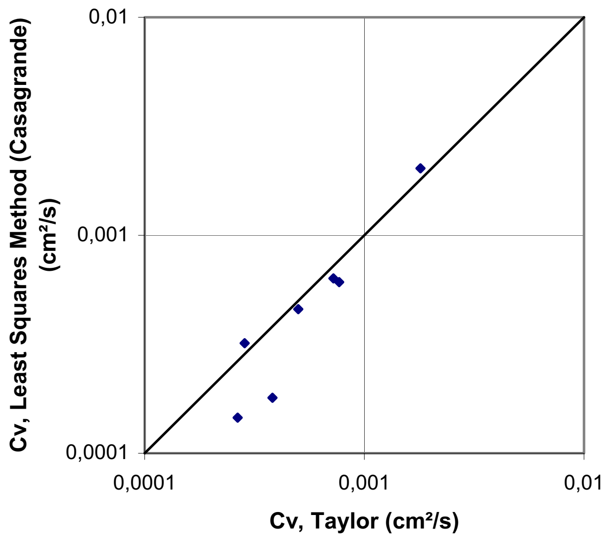

Figure 4. Values of coefficient of consolidation obtained from Least Squares method and Taylor's method.

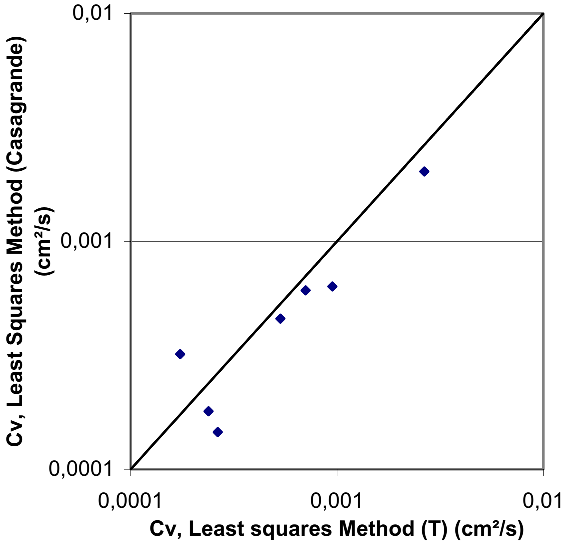

Figure 4. Comparison of C_V obtained using Least Squares method with Casagrande and Taylor starting points.

Table 2. Classification of methods in increasing order of squares of residuals.

	Weights are zero when $U_{avg, exp}$ greater than		
rank	60%	90%	100%
1	Least Squares	Least Squares	Least Squares (Casagrande)
2	Taylor	Taylor	Taylor
3	Linear segment of the t½ method	Casagrande	Casagrande
4	Casagrande	Linear segment of the t½ method	Linear segment of the t½ method
5	Inflection Point Method	Inflection Point Method	Inflection Point Method
6	Rectangular hyperbola	Rectangular hyperbola	Rectangular hyperbola

For the purpose of comparison on a same scale, the sum of the squares of residuals between the theoretical and measured average degree of consolidation have been calculated. To achieve this, the C_V obtained by each method was used to calculate the estimated average degree of consolidation. The experimental values have been calculated using the same procedure as for the Least Squares method, but using the corresponding values of compression available from each method, for example d_{70} can be obtained from the Inflection Point Method. In the evaluation of the squares of the residuals, the end of primary compression has been taken up to 60% and 90%. Table 2 shows the various methods classified in increasing order of the sum of the squares of residuals. For the weight of 100%, the Least Squares method was computed from initial parameters obtained from the Casagrande method.

The Least squares method predominates in terms of the least sum of the squares of residuals, which infers a most successful fitting of the Terzaghi theoretical T-U_{avg} and experimental curves. The shift between the Casagrande method and the Linear segment of the t½ method from an average consolidation of 60% and 90% is due to the weight of the system. This means that between 60% and 90%, the sum of the squares of the residuals increase to a greater extent in the linear segment of the t½ method than in the Casagrande method because the end of primary compression in the first method is at 60% compared to 100% in Casagrande's method.

3. CONCLUSION

The coefficient of consolidation is a useful tool in engineering for the prediction of the rate of settlement in soils. In this paper, some methods of finding this coefficient from laboratory oedometer tests have been reviewed and compared to an alternative method which is the Least Squares method. Its principle relies on the fitting between theoretical and experimental values of the average degree of consolidation. An unbiased estimator is used to compute values of compression d_0 and d_{90} or d_{100} as well as the value of C_V based on minimising the squares of the residuals between the average degree of consolidation values. The coefficients determined from this method are in good agreement with the results found from the other methods and provide an excellent numerical tool for this purpose. It can therefore be concluded that the Least Squares method is effective in the determination of the coefficient of consolidation. The method is also found to give the 'best fit' to the experimental data.

4. REFERENCES

Bowles, J. E., 1992. Engineering Properties of soils and their measurement. 4th Edition, McGraw Hill.

Casagrande, A. & Fadum 1940 Notes on soil testing for Engineering Purposes. Harvard Soil Mechanics No 8.

Feng, T.W. and Lee, Y.J., 2001. Coefficient of Consolidation from the linear segment of the t½ curve. Canadian Geotechnical Journal 2001, **38:** 901–909.

Holtz, R.D. and Kovacs, W.D., 1991. Introduction à la Géotechnique. Éditions de l'école polytechnique.

Liu, C. and Evett, J. B. 2003. Soil properties, Testing, Measurement and their Evaluation. 5th edition, Prentice Hall.

Mesri, G., Feng, T.W., and Shahien M. 1999. Coefficient of consolidation by Inflection Point method. Journal of geotechnical and geoenvironmental engineering, **125**(8): 716-718.

Robitaille, V. et Tremblay, D. 1997. Mécanique des sols, théorie et pratique. Éditions Modulo.
Sridharan, A., Murthy, N. S., Prakash, K. 1987 Rectangular Hyperbola method of consolidation analysis. Géotechnique **37** (3) : 355-368
Taylor, D. W. 1948. Fundamentals of soil mechanics. John Wiley and Sons, New York.
Wray, W. K. 1986. Measuring Engineering Properties of Soils. Prentice Hall.