# COMPARING SLOPE STABILITY ANALYSIS BASED ON LINEAR ELASTIC OR ELASTO PLASTIC STRESSES USING DYNAMIC PROGRAMMING TECHNIQUES

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ABSTRACT
Conventional limit equilibrium methods of slope stability analysis are usually combined with restrictions to the shape of the slip surface to overcome indeterminacy in the formulation. The Dynamic Programming Method (DPM) of slope stability represents an important breakthrough in that additional physics are incorporated rather than restrictions. The stress strain behavior of the soil is included in the analysis through the combination of the Dynamic Programming (DP) optimization technique and a finite element stress analysis. This paper presents a series of examples comparing results between slope stability analysis based on stresses generated from either a linear elastic or elasto-plastic stress analysis. Stability analyses based on the two constitutive models are compared using the shape, location, and factor of safety of the critical slip surface. The comparison shows no significant deviation between results based on either constitutive model when factors of safety are equal to or greater than one.

RÉSUMÉ
Les méthodes conventionnelles d'analyse de stabilité à l'équilibre limite sont généralement associées à des limitations sur la forme de la surface de glissement afin d'éviter des indéterminations dans la formulation. La Méthode de Programmation Dynamique (Dynamic Programming Method, DPM) de stabilité des pentes représente un important progrès puisqu'elle fait davantage appel à la physique plutôt qu'à des restrictions. Le comportement contrainte-déformation du sol est pris en compte dans l'analyse par l'utilisation de la technique d'optimisation de la Programmation Dynamique (Dynamic Programming, DP) et d'une analyse des contraintes par éléments finis. Cet article présente une série d'exemples comparant les résultats d'analyses de stabilité de pentes basées sur des contraintes déduites d'analyses linéaires-élastiques ou élasto-plastiques. Les analyses de stabilité basées sur les deux modèles constitutifs sont comparées sur la base de la forme, de la position et du coefficient de sécurité de la surface de glissement critique. La comparaison ne montre pas de différence significative entre les résultats quand les coefficients de sécurité sont égaux ou supérieurs à l'unité.

## 1. INTRODUCTION

Conventional limit equilibrium slope stability methods require assumptions regarding stresses within the slope and the shapes of potential slip surfaces to render the problem determinate. These assumptions have been overcome with the use of the Dynamic Programming Method (DPM) in combination with stresses from a finite element analysis. However, slope stability calculations based on stresses from finite element analysis have not become popular for slope stability studies due to intense computational requirements and difficulties in assessing the stress versus strain characteristics of the soil (Scoular 1997). If the DPM of analysis is to be adopted in practice, these issues must be addressed and continuing verification of the method must be provided.

The focus of this study is to address the above concerns by providing additional verification of the DPM and guidance as to the appropriate constitutive model to use in stability calculations. Verification of the DPM is provided through the comparison of a vertical cut analysis with Taylor's stability charts. Guidance on appropriate constitutive models is provided through a comparison of stability results based on either Linear Elastic (LE) or Elasto-Plastic (EP) stresses. The comparison includes a series of stable and failing slopes. It is of interest to determine if stresses from a plastic analysis are required in order to correctly determine the shape, location, and factor of safety of critical slip surfaces for slopes at or near failure.

## 2. BACKGROUND

Much research has been focused on developing techniques to overcome assumptions used in limit equilibrium methods. Variational calculus was one of the first of such techniques. The calculus of variations provides a mathematical procedure to find the shape of an extremal, a curve that maximizes or minimizes the value of an integral along that line (De Josselin De Jong 1980). The benefit of such a method is that the factor of safety is determined without any prior assumptions regarding the shape or location of the critical slip surface. While the calculus of variations was popular with several researchers, it was shown to contain a degeneration (De Josselin De Jong 1981). The degeneration was related to the non-existence of unique derivatives within the formulation.

The DPM, in combination with Spencer's (1967) assumptions, was first applied to slope stability problems by Baker (1980). It was used to overcome assumptions regarding the shape of potential slip surfaces and degenerations reported with variational calculus. Baker notes that while the DPM is similar in concept to the calculus of variations it does not require the existence and uniqueness of derivatives to determine the critical slip surface. Instead, the minimization is completed numerically through the direct comparison of values. While numerical methods may have been laborious in the past, they are much less of a problem with the high speed computers readily available today.

The formulation of the DPM in combination with stresses from a finite element analysis was first developed by Yamagami and Ueta (1988b). The purpose of the study was to employ limit equilibrium methods to yield an overall factor of safety while accounting for the constitutive relationship and initial stress state of the soil using stresses from a finite element analysis. The benefit of determining the shape of the slip surface without assumptions was also realized. Further verification has been provided by testing the technique over a wide range of slope conditions using a large parametric study (Pham 2002). The technique has also been applied to the analysis of transient embankment stability by Gitirana and Fredlund (2003).

## 3. VERIFICATION

### 3.1 Vertical Cut analysis

The $\phi$-Circle Method was proposed by Professors Glennon Gilboy and Arthur Casagrande with the hope that a completely graphical solution method may be developed to solve for the stability of a homogeneous slope. Applied to any circular slip surface, the result is a vector whose length represents the quantity $2c/\gamma$, where $c$ is the cohesion and $\gamma$ is the unit weight required for equilibrium (Taylor 1937). Dividing the length of the vector by the height of the slope ($H$), resulted in an abstract number that could describe the equilibrium conditions for a slope of any height for a given slope and friction angle, $2c/\gamma H$. Taylor (1937) modified the form of this abstract number resulting in the following dimensionless expression called the "Stability Number" with $F$ representing the factor of safety.

$$\frac{c}{F\gamma H} \qquad [1]$$

The traditional factor of safety equation becomes independent of the normal stress when $\phi'$ is equal to zero. If the linear elastic constitutive model is used, the calculation of the factor of safety using the DPM should be independent of Young's Modulus and Poisson's ratio. These realizations allow for the comparison of the DPM with the stability charts developed by Taylor (1937) which are also independent of stress. The reliability of the DPM will be tested by comparing the factor of safety obtained for a strictly cohesive vertical cut with Taylor's stability chart.

A slope with cohesion equal to 30 kPa, internal angle of friction equal to zero, and a unit weight of $18 kN/m^3$ was used to compare the DPM with Taylor's stability chart. Assuming a factor of safety of one, Taylor's stability number predicts the critical height for a vertical cut with the noted properties to be 6.4m. A stress analysis is completed for a vertical cut with this height using the linear elastic constitutive model. Two values of Poisson's Ratio (0.48 and 0.33) and Young's Modulus (20,000 kPa and 100,000 kPa) are chosen to test if the DPM is independent of stress for the $\phi'$=0 analysis.

## 4. COMPARISON OF SLOPE STABILITY RESULTS BASED ON LINEAR ELASTIC OR ELASTO-PLASTIC STRESS ANALYSIS

The purpose of this comparison is to address issues related to basing a slope stability analysis on stresses generated from a numerical analysis. More specifically, investigating the use of linear elastic stresses in slopes where the factor of safety is low enough to allow overstressing to occur. In such slopes, a constitutive model that accounts for yielding may be required in order to correctly calculate the factor of safety and determine the shape and location of the critical slip surface. However, use of plastic constitutive models is more involved causing computing times to increase. Therefore, it is necessary to determine if a plastic analysis is in fact required.

### 4.1 Scope

A variety of increasingly complex problems are chosen to ensure that the comparison will encompass a small range of typical conditions. The range of problems includes homogeneous slopes and multi-layered slopes with various Poisson ratios, pore pressure conditions and slope angles. Space does not permit the presentation of every model so representative results are chosen that best illustrate the observed behaviour for each condition.

The FLAC stress analysis software package is chosen to complete the comparison. FLAC is chosen based on three requirements. The software must include an elasto-plastic constitutive model, a linear elastic constitutive model, and must have the ability to incorporate the computed stress data into the DP slope stability analysis. It should also be noted that the finite difference solution technique employed by FLAC results in similar stress fields when compared with finite element results for the range of slopes in this study.

#### 4.1.1 Homogenous Slopes

Ideally, the comparison between the LE and EP models will be carried out over a range of safety factors.

Beginning with safety factors for which the majority of the slope behaves in an elastic manner progressing to safety factors where the slope experiences yielding. Soil properties are chosen such that three factors of safety will result including ~1.3, ~1.0, and < 1.0. Factors of safety of 1.3 and 1.0 result in slopes that are mostly elastic with some yielding and slopes that are at failure, respectively. The majority of engineering design is completed using factors of safety greater than 1.2 while successful back analysis requires a factor of safety of 1.0. Factors of safety less than 1.0 are impossible in reality but are included for discussion purposes.

Table 1. Homogeneous slope soil properties

| $\gamma$ (kN/m$^3$) | $\mu$ | $c'$ (kPa) | $\phi'$ |
|---|---|---|---|
| 18 | 0.48 | 20 | 10 |
| | | 17 | 7 |
| | | 15 | 5 |
| 18 | 0.40 | 20 | 10 |
| | | 17 | 7 |
| | | 15 | 5 |
| 18 | 0.33 | 20 | 10 |
| | | 17 | 7 |
| | | 15 | 5 |
| 18 | 0.20 | 20 | 10 |
| | | 17 | 7 |
| | | 15 | 5 |

The range of soil properties used in the homogeneous slopes is shown in Table 1. The effective cohesion ($c'$) and effective friction angle ($\phi'$) are chosen such that a factor of safety of ~1.3 results, given a unit weight of 18 kN/m$^3$ and a 2:1 slope angle. The slope is brought to failure by decreasing the cohesion and friction of the material by equal increments resulting in factors of safety of ~1.0 and <1.0 respectively. A set of analysis was also completed bringing the slope to failure by increasing the unit weight of the material. Using cohesion equal to 20 kPa and friction angle equal to 10° the desired range of safety factors was achieved by using unit weights of 18 kN/m$^3$, 30 kN/m$^3$, and 50 kN/m$^3$. Bringing the slope to failure by increasing the unit weight produced similar results therefore only the strength reduction results are presented.

Four values of Poisson's ratio are chosen including 0.48, 0.40, 0.33, and 0.2. Varying Poisson's ratio in a LE analysis is equivalent to setting different in-situ stress conditions. The range of $K_o$ values achieved using the above values for Poisson's ratio include 0.25, 0.5, 0.7, and ~1.0. These values are calculated according to the following relationship, $K_o = \mu / 1\text{-}\mu$. This relationship is only a guide, as stress conditions generated in a slope vary with depth. It should also be noted that Young's Modulus was held constant at 20,000 *kPa* for all homogeneous analysis.

4.1.2 Wet & Submerged Slopes

Water is included for the case of Poisson's ratio equal to 0.48, cohesion equal to 20 kPa, friction angle equal to 10° and slope angle of 2:1. Two pore-water conditions are generated including a wet slope condition where the water table is drawn down to the toe of the slope and a submerged slope condition where the water is two meters deep at the toe of the slope.

4.1.3 Slope Angle

Three slope angles are chosen including 3:1, 2:1 and 1:1. These slope angles are applied to the case where Poisson's ratio is 0.48 and the unit weight is 18 kN/m$^3$. Given these conditions the factor of safety ranges from ~1.6 to 1.0.

4.1.4 Mult-Layered Slopes

Multi-layered slopes are included to compare conditions where a contrast in Young's Modulus exists and a weak soil layer is included. Two conditions are considered including a 2-layer slope and a 3-layer slope under dry and wet conditions.

## 5. RESULTS

### 5.1 Vertical Cut

The factor of safety calculated by the DPM is virtually the same as that predicted by Taylor's stability chart. Focusing on the first two slip surfaces listed in Figure 1, the factor of safety calculated using a constant Poisson's ratio is exactly the same whether the Young's Modulus is 20,000 kPa or 100,000 kPa. The slip surfaces for the first two cases also plot one on top of the other. Therefore, the factor of safety and the critical slip surface calculated using the DPM method is independent of Young's Modulus.

Comparing the results using a Poisson's ratio of 0.48 to 0.33 yields a slight difference in the factor of safety and the location of the critical slip surface. As noted by (Gitirana and Fredlund 2003), the overall effect of the value of $\mu$ results in higher $\sigma_x/\sigma_y$ ratios inside the embankment for larger values of $\mu$. The larger values of $\sigma x/\sigma y$ inside the embankment drive the slip surface to shallower regions where the lower $\sigma_x/\sigma_y$ ratios occur. While the effect is almost negligible in this case, the slip surface for $\mu$ equal to 0.33 is slightly deeper than for the 0.48 case. In this comparison it appears that various in-situ stress states result in only minor differences.

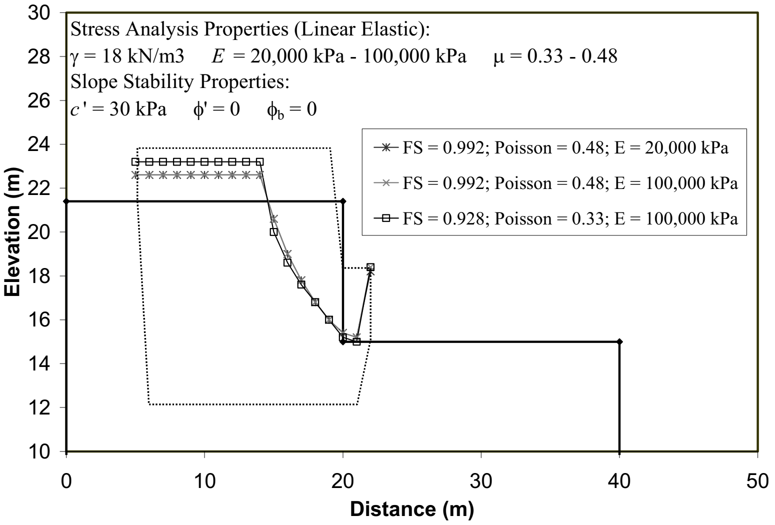


Figure 1. Vertical Cut Comparison

5.2 Homogeneous Slope

The results for the case of Poisson's ratio equal to 0.48 are used to illustrate the observed behaviour for all dry homogeneous slopes. There are three figures, one for each factor of safety including ~1.3, ~1.0, and <1.0. It should be noted that the legend for each slip surface records whether the analysis was elastic or elasto-plastic and the calculated factor of safety. The factor of safety calculated by the Morgenstern Price (MP) method using a half sine function is also included for comparison. One summary plot at the end of the section shows the relationship between the initial $K_o$ conditions and the factor of safety.

Figure 2, Figure 3 and Figure 4 illustrate the difference between slope stability analysis based LE or EP stress analysis. The shape and location of the critical slip surfaces are very similar for conditions where the factor of safety is equal to or greater than one. The calculated factors of safety for these conditions are also very similar. The following reasoning explains why the differences between these two analyses are very small.

The distribution of stress computed by any stress analysis is governed mainly by the geometry of the problem, the boundary conditions and the soil properties. Therefore, it is reasonable to expect that critical areas of high stress will develop in similar locations within a slope using either the LE or EP constitutive model. This can be verified by using extremely high strength in an EP stress analysis and observing that the stress distribution is the same as the equivalent LE stress analysis.

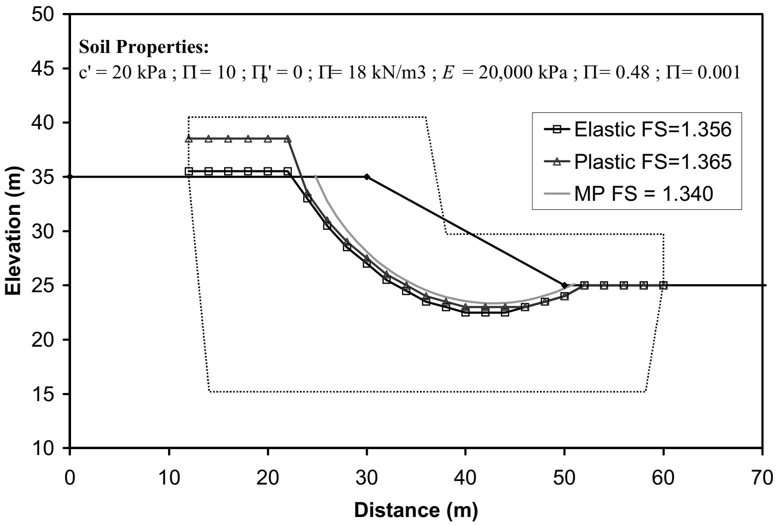


Figure 2. Dry Homogeneous Slope, FS~1.3

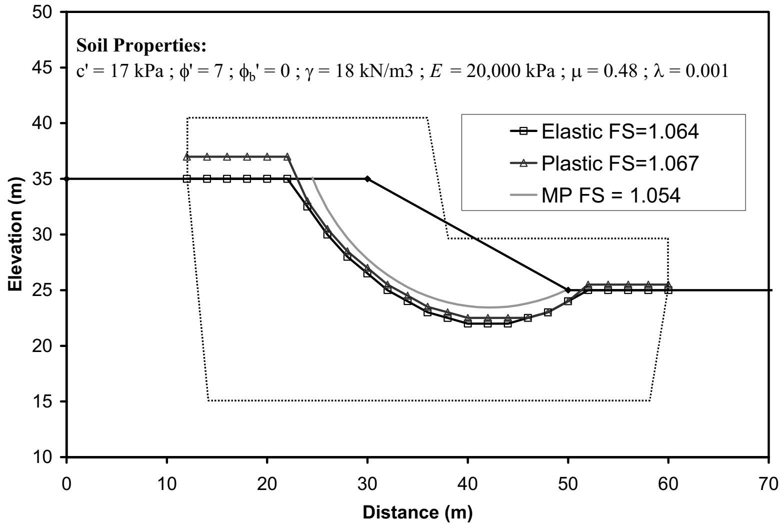


Figure 3. Dry Homogeneous Slope, FS~1.0

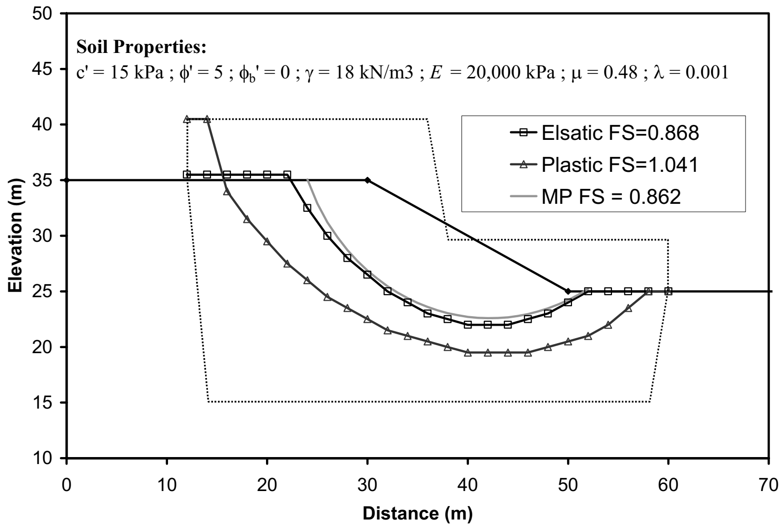


Figure 4. Dry Homogeneous Slope, FS<1.0

However, in most cases the EP stresses are only allowed to increase to a certain level before excess stresses are redistributed to surrounding soil. As shown in Figure 2 and Figure 3, this does not cause large differences in the location of the critical slip surface when the factor of safety is equal to or greater than 1.0. Excess stresses are redistributed to adjacent areas of high stiffness and lower mobilized strength The DP stability analysis searches for the path of lowest mobilized strength. Therefore, it is not surprising that the DP search determines the critical slip surface to be in similar locations using either constitutive model. This has also been verified by comparing displacement vectors determined from EP stress analysis with the critical slip surface determined from DP stability analysis. If additional load is added to the slope or the strength of the slope is decreased, large displacements occur. Additional Stresses are redistributed to surrounding soil increasing the amount of failed soil. This causes large differences in the location of the critical slip surface determined from LE or EP stresses as shown in Figure 4.

The discussion above is also useful for explaining differences observed in the distribution of the factor of safety but overall agreement in the average factor safety. If strength properties used in an EP model are such that a factor of safety of 1.0 will result, the factor of safety computed for individual line segments along the critical slip surface will be approximately the same. If DP stability analysis is completed using the equivalent LE stresses, the distribution of the factor of safety is highly variable and in some cases exists below 1.0. This difference is also due to the redistribution of stresses that occurs in an EP analysis. Although there are differences in the distribution of the factor of safety along the critical slip surface, agreement in the overall factor of safety results from three conditions: 1) the same geometry, boundary conditions and elastic constants are used to generate the stress state, 2) the same strength properties are used to perform the stability analysis and 3) the calculated factor of safety for the slope is equal to or greater than 1.0.

The same trend is observed when the initial stress state is changed. There remain only small differences in stability analysis based on either LE of EP stress analysis. However, it should be noted that changing the initial stress state causes subtle variations in the shape of the critical slip surface from both analysis. The slip surface becomes less circular with decreasing Poisson’s ratio (i.e. decreasing $K_o$). The effects are more pronounced near the crest of the slope. Figure 5 illustrates the relationship between factor of safety and initial stress state computed using LE stresses. The plot shows that there is virtually no change in the factor of safety as $K_o$ is varied from 0.3 to ~1.

The remaining examples have been included to examine if similar results will be observed for more complex slopes.

5.3 Wet & Submerged Slope

Figure 6 shows the comparison of LE wet (Elastic-W), EP wet (Plastic-W), LE submerged (Elastic-S), and EP submerged (Plastic-S) results.

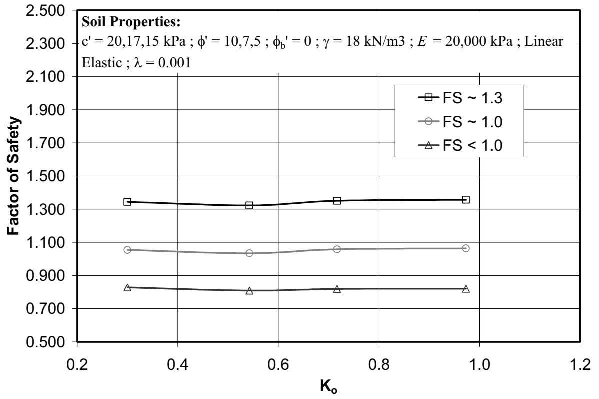


Figure 5. Effect of initial stress state on the factor of safety.

As anticipated the critical slip surface rises and the factor of safety increases slightly as the slope is submerged with water. This occurs due to the added pressure of the standing water at the toe of the slope. It can be seen that the shape and location of the critical slip surfaces are again quite similar for both the wet and submerged conditions. From Figure 6, the dark lines correspond to the wet condition and the light lines correspond to the submerged condition.

5.4 Slope Angle

As expected, the factor of safety decreases as the slope angle becomes steeper. The results also show the same similarities between the shape and location of the critical slip surface when using either LE or EP stresses.

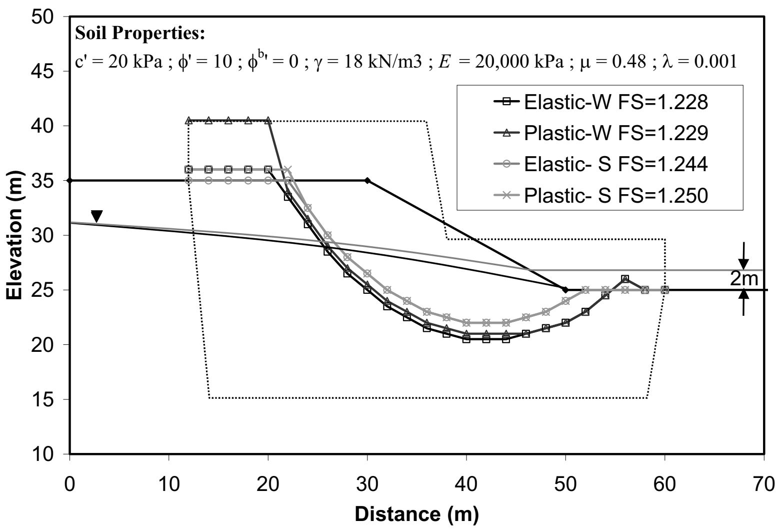


Figure 6. Wet & Submerged Slope

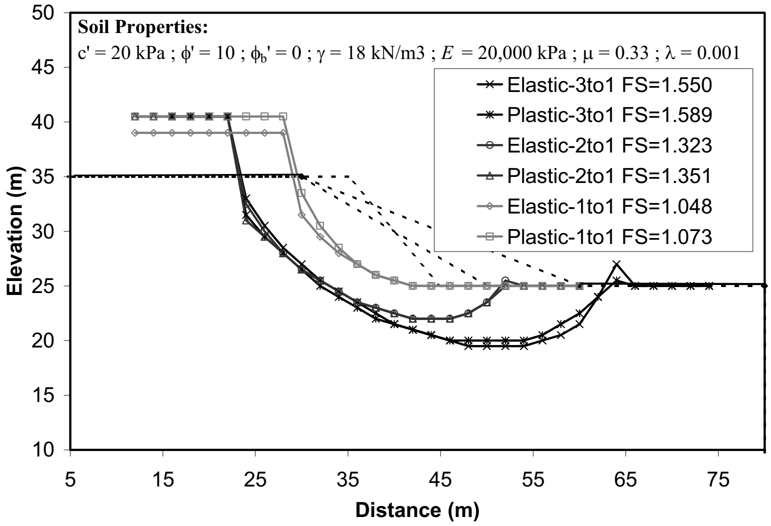


Figure 7. Slope Angle

## 5.5 Multi-Layer Slopes

### 5.5.1 Two Layer Slope

Figure 8 illustrates the results for the 2-Layer slope at a factor of safety of ~1.0 and dry conditions. Only small differences exist between the location of the critical slip surface and the calculated factor of safety. The two methods differed by similar amounts when water was included.

### 5.5.2 Three Layer Slope

The three layer slope models a condition of a much larger stiffness contrast, as well as the inclusion of a weak layer. The results from the dry condition are shown in Figure 9. While these results show almost no deviation, it should be noted that a limitation was encountered in this analysis. The thickness of the weak layer was originally 1m thick. The results from this condition showed larger deviations near the toe of the slope and through the weak layer. This deviation was due to the search grid density relative to the dimensions of features within the slope. When the thickness of the weak layer is increased to 2m, the results calculated are as shown in Figure 9. More testing is required to determine the optimum grid density relative to the size of important features within the slope.

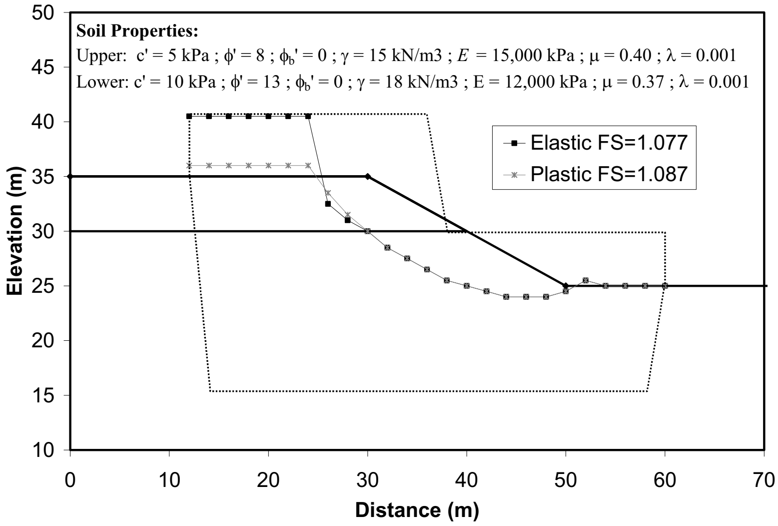


Figure 8. Two Layer Slope

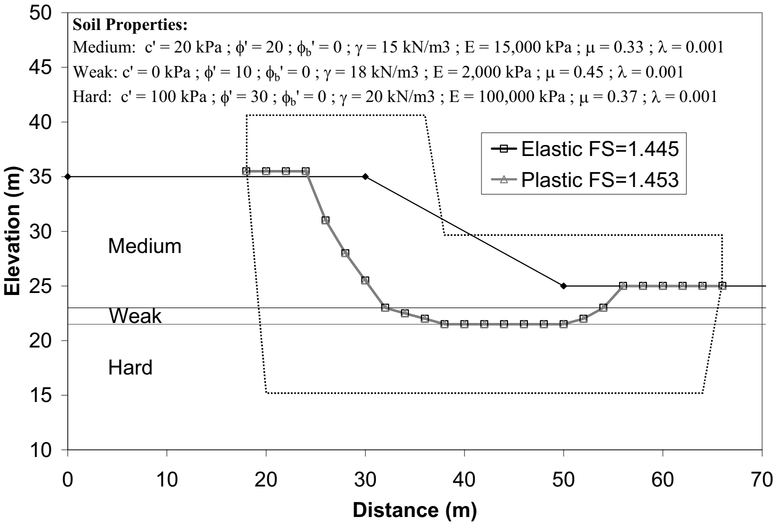


Figure 9. Three Layer Slope

## 6. CONCLUSIONS

The reliability of the DPM is confirmed in the vertical cut analysis. The DPM predicted the same factor of safety as Taylor's stability charts for a vertical strictly cohesive cut. It was also demonstrated that the factor of safety and slip surface were relatively unaltered by Young's Modulus and only minor changes resulted from changing Poisson's ratio.

The differences between stability analyses based on LE or EP stresses are also presented. The largest differences are shown to occur in the fabricated condition where the factor of safety is less than one. Only small differences were found when the slope was modelled under more realistic conditions where the factor of safety is equal to or greater than 1.0. It would seem then that LE stress analysis is sufficient to calculate the overall stability of a slope. Due to the various conditions considered, it is reasonable to conclude that elastic stress analysis is adequate for a large range of slope conditions frequently encountered in practice. This is of benefit as elastic stress analysis is less involved requiring much less computer time. However, more research is required to determine if this conclusion will hold true for every possible slope condition encountered in practice.

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