A MODEL TO PREDICT THE UNSATURATED HYDRAULIC CONDUCTIVITY FROM BASIC SOIL PROPERTIES

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ABSTRACT
The relative hydraulic conductivity k_r of unsaturated soils is typically obtained from their water retention curve (WRC). In this paper, the modified Kovács (MK) model developed to predict the WRC is combined with the Mualem model to predict the k_r function using basic geotechnical properties of granular soils. The ensuing equations, which have been implemented into MATLAB, have been validated against existing solutions and against testing data. It is shown here that the predictive results show a good agreement with the experimental results obtained from tests performed by the Authors and taken from the literature. The applicability of the proposed approach is briefly discussed.

RÉSUMÉ
La conductivité hydraulique relative k_r de sols non saturés peut être obtenue à partir de leur courbe de rétention d'eau (CRE). Dans cet article, le modèle de Kovács modifié (MK) développé pour prédire la CRE est combiné avec le modèle de Mualem pour prédire la fonction k_r en utilisant les propriétés géotechniques de base de sols granulaires. Les équations qui en résultent ont été résolues avec MATLAB et validées à l'aide de solutions existantes et de données d'essais. On montre ici que les résultats des prédictions concordent bien avec les résultats expérimentaux obtenus des tests de drainage effectués par les auteurs et avec des résultats tirés de la littérature. L'applicabilité de l'approche proposée est brièvement discutée.

1. INTRODUCTION

Richards' (1931)'equation is commonly used to represent water flow in the vadose zone. This equation is typically expressed as function of the volumetric water content θ (L^3/L^3), hydraulic head H [L], elevation z (L), time t (T), and unsaturated hydraulic conductivity function k_u (L/T) expressed as a function of suction ψ or of θ. To solve this equation, the k_u function thus needs to be defined.

Various techniques have been developed to measure, in the laboratory or in the field, the unsaturated hydraulic conductivity k_u (e.g. Klute and Dirksen 1989). These techniques can be time-consuming and expensive. Furthermore, the number of measurements required to adequately characterize an area can become prohibitive. It is thus helpful to have means to estimate, in a simple and practical manner, the value of k_u. For that purpose, the water retention curve WRC is often used (e.g. Green and Corey 1971; Mualem 1976, 1986; Leong and Rahardjo, 1997). The WRC can nevertheless be cumbersome to obtain.

This article deals with the extension of the modified Kovács (1981) model (MK), developed to predict the WRC (Aubertin et al. 1998, 2003; Mbonimpa et al. 2000), to estimate the relative unsaturated hydraulic conductivity function k_r of granular soils. The predictions require basic geotechnical properties including the effective diameter D_{10}, uniformity coefficient C_U, and void ratio e. The proposed approach is evaluated by comparing predicted values to measured data obtained from free drainage column tests performed by the Authors and others taken from the literature.

2. THE MK MODEL TO PREDICT THE WRC

The modified Kovács (MK) model predicts the WRC of non compressible materials, under drainage conditions, using basic geotechnical properties (Aubertin et. al. 2003). The MK model considers that water is held by two types of forces, i.e. capillary forces responsible for a capillary saturation S_c and adhesive forces causing saturation by adhesion S_a (Kovács 1981). The S_c component equation is obtained from a pore size distribution function, while the equation for S_a is given by an interaction law between grain surface and water dipoles.

The MK model equations can be written as follows, for the degree of saturation S_r:

$$S_r = \frac{\theta}{n} = S_c + S_a^*(1 - S_c) \quad [1]$$

with

$$S_c = 1 - \left[(h_{co}/\psi)^2 + 1 \right]^m \exp\left[-m (h_{co}/\psi)^2 \right] \quad [2]$$

$$S_a^* = 1 - \langle 1 - S_a \rangle \quad [3]$$

and

$$S_a = a_c\, C_\psi \frac{\left(h_{co}/\psi_n\right)^{2/3}}{e^{1/3}\left(\psi/\psi_n\right)^{1/6}} \quad [4]$$

$$C_\psi = 1 - \frac{\ln(1+\psi/\psi_r)}{\ln(1+\psi_o/\psi_r)} \quad [5]$$

Equation 1 expresses the total degree of saturation S_r (=θ/n, where n is the porosity) by combining the capillary and adhesion components S_c and S_a. A truncated value of S_a (i.e. S_a^*) is used to make sure that the adhesion degree of saturation does not exceed unity at low suction. In equation 3, $\langle\ \rangle$ represents the Macauley brackets ($\langle y\rangle$=0.5(y+ $|y|$)); for $S_a \geq 1$, $S_a^* = 1$, and for $S_a < 1$, $S_a^*=S_a$. In equations 2 and 4, h_{co} [L] is the equivalent capillary height (defined below), ψ [L] is the matric suction head; m (-) is a pore size distribution coefficient, a_c (-) is the adhesion coefficient, e is the void ratio, and ψ_n is a normalization parameter introduced for unit consistencies (ψ_n =1 cm when ψ is given in cm, corresponding to a suction of 10^{-3} atmosphere). Parameter C_ψ (equation 5) forces the water content to zero when ψ reaches a limit imposed by thermodynamic equilibrium (θ = 0 at $\psi = \psi_0 = 10^7$ cm of water, corresponding approximately to complete dryness; Fredlund and Xing 1994). In this equation, ψ_r represents the suction at residual water content, which depends on basic soil properties (as is the case with h_{co}).

The equivalent capillary height h_{co} is related to an equivalent pore diameter and it can be estimated using the following relationship:

$$h_{co} = \frac{0.75}{[1.17\log(C_U)+1]\ e D_{10}} \quad [6]$$

where D_{10} is the effective diameter (diameter corresponding to 10 % passing on the cumulative grain-size distribution curve), and C_U is the uniformity coefficient (=D_{60}/D_{10}); h_{co} and D_{10} are expressed in cm.

The value of m, a_c and ψ_r can also be related to basic properties. For granular (non cohesive, low plasticity) soils, analyses have indicated that the value of the pore-size distribution parameter m can often be closely approximated by $m \cong 1/C_U$, while the adhesion coefficient a_c is approximately constant at 0.01 (when head parameters are in cm). The residual suction ψ_r can be calculated as:

$$\psi_r = 0.86\ h_{co}^{1.2} \quad [7]$$

The MK model provides good estimates of drainage WRC for low plasticity soils and similar materials (Mbonimpa et al. 2000, Aubertin et al. 2003). This model is used here to obtain the k_r function.

3. THE MUALEM k_r-MODEL

A number of relationships have been developed in recent years to relate k_u with other materials properties (e.g Vereecken et al. 1992; Elsenbeer 2001; Wösten et al. 2001). In doing so, physically-based models should be preferred to obtain more flexible predictive methods.

The following definition is used with most models:

$$k_u = k_r k_s \quad [8]$$

where k_s [LT^{-1}] and k_r [-] are the saturated and relative hydraulic conductivities, respectively. It is assumed here that k_s is known for the soil at hand, and that the function k_r can be derived from the WRC using statistical methods. A few formulations have been proposed in that regard, following the work of Childs and Collis-Georges (1950). The most often used models are probably those of Burdine (1953) and Mualem (1976), which can expressed by the following formulation:

$$k_r(\theta) = \theta_e^{\alpha}\left(\frac{\int_{\theta_r}^{\theta}\psi^{-\beta}dy}{\int_{\theta_r}^{\theta_s}\psi^{-\beta}dy}\right)^{\delta} \quad [9]$$

with

$$\theta_e = \frac{\theta-\theta_r}{\theta_s-\theta_r} \quad [10]$$

In these equations, θ_e, θ, θ_r and θ_s represent the reduced (dimensionless), actual, residual, and saturated volumetric water contents [L^3L^{-3}]; y [L^3L^{-3}] is an integration variable associated to θ. Empirical parameters α, β and δ take the values α = 2, β = 2 and δ= 1 for the Burdine (1953) formulation, and of α = 0.5, β= 1 and δ = 2 for the Mualem (1976) formulation. These models lead to k_r = 0 when $\theta \leq \theta_r$ (or $\psi \geq \psi_r$, where ψ_r is the residual suction associated θ_r).

In the next section, the Mualem formulation is combined with the modified Kovács (MK) model to estimate k_r.

4. THE MATLAB-k_r-MK CODE

Equation 9 can be used to calculate the relative hydraulic conductivity k_r, when the WRC is defined as $\psi(\theta)$. The integrations are performed with the volumetric water content (θ). The MK model function $\theta(\psi)$ defined with equations 1 to 7 admits no reciprocal function $\psi(\theta)$, so a change of variable is required to make ψ the integration variable. In this case, the Mualem (1976) model is written in the following form:

$$k_r(\psi) = \theta_e^{\alpha} \left(\frac{\int_{\psi}^{\psi_r} \theta' \psi^{-\beta} dv}{\int_{\psi_s}^{\psi_r} \theta' \psi^{-\beta} dv} \right)^{\delta} \qquad [11]$$

In this equation, ψ_s is the suction associated to θ_s, and v is an integration variable representing the suction. The parameter ψ_s is given a small finite value (i.e. ψ_s = 0.01 cm) to avoid mathematical indetermination (instead of ψ_s = 0). Function θ' represents the derivative of the volumetric water content function $\theta(\psi)$ ($\theta'=d\theta/d\psi$), calculated from equation 1. For ψ smaller than the air entry value (AEV), θ is equal to the porosity n (=θ_s) and θ' = 0. For ψ > AEV, θ' becomes:

$$\theta' = n\left[S'_c(1 - S_a) + S'_a(1 - S_c)\right] \qquad [12]$$

where S'_c and S'_a are the derivative of the S_c (eq. 2) and S_a (eq. 4) functions.

Solving equation 11 for the MK model requires a numerical treatment, so the equations have been implemented into MATLAB (Hanselman and Littlefield 1997). To obtain $k_r(\psi)$, the user needs the effective diameter D_{10} (in cm), the uniformity coefficient C_U (-), and the void ratio e (-). The calculations are made with the MAT-k_r-MK code for incremental values of ψ ($\leq \psi_r$).

Values for k_r obtained with the proposed numerical solution (MAT-k_r-MK code) have been compared to results provided by the RETC code (van Genuchten et al. 1991; Yates et al. 1992). For this comparative assessment, the van Genuchten (1980) model has been used. The results obtained (not shown here) indicate that the two calculation approaches provide very close $k_r(\psi)$ curves for similar conditions.

5. SAMPLE APPLICATIONS

The proposed approach to estimate k_r is evaluated by comparing predicted and measured values. The Authors' own results and some data taken from the literature are used for these comparisons. Table 1 gives the basic geotechnical properties for the granular materials considered in this study. The experimental WRC and the k_u data were obtained using different techniques.

Table 1. Basic properties of the granular soils.

Sample	n (-)	θ_s (-)	D_{10} (cm)	C_U (-)	k_s (cm/s)
Sand[1]	0.333	-	0.0110	3.7	5.45×10^{-3}
Fontaine-bleau[2]	0.351	-	0.0218	1.3	5.00×10^{-3}
El Oued[2]	0.320	-	0.0122	2.0	1.70×10^{-3}
Code 1460[3]	0.297	0.261	0.0224	2.3	2.91×10^{-3}
Code 2221[3]	0.328	0.314	0.0054	8.7	1.45×10^{-2}
Code 1461[3]	0.373	0.362	0.0224	2.3	2.31×10^{-2}
Code 1463[3]	0.399	0.388	0.0144	2.4	8.01×10^{-3}
Code 4660[3]	0.460	-	0.0068	5.2	7.24×10^{-3}

[1]Authors'data
[2]Data from Soeiro (1964)
[3]Data from UNSODA (Leij et al. 1996, Nemes et al. 2001)

5.1 Authors' experimental data

Free drainage tests have been carried out on initially saturated sand placed in a vertical column made of acrylic cylinder with an inside diameter of 15.4 cm. The total column height is about 150 cm. The equipment and configuration are shown in Figure 1 (adapted from Bédard 2003). The column was instrumented with TDR probes and tensiometers (connected to pressure transducers) to measure the water content θ and water pressure u_w ($ML^{-1}T^{-2}$) (or suction head ψ), respectively. The instruments were linked to a control panel and data acquisition system.

The sample was flooded with distilled, deaired water using an upward hydraulic gradient. Full saturation is attained when the tensiometer readings stabilize and correspond in height to the water column above the tensiometer position. The water table was lowered from the sand surface to the base of the column, and the water was allowed to drain out freely by opening a valve. Volumetric water contents and water pressures were recorded at different positions and time intervals. At the beginning of the experiment, when the pressure and water content changes were rapid, measurements were recorded at small time intervals; as the changes became more gradual, measurements were recorded at larger time intervals.

After reaching equilibrium following drainage (when the bottom flux is zero, and ψ and θ are constant), the sample was "re-saturated" as described above. The next drainage test could then begin (Bédard 2003). The testing results shown here included four tests carried out on the sand in this manner.

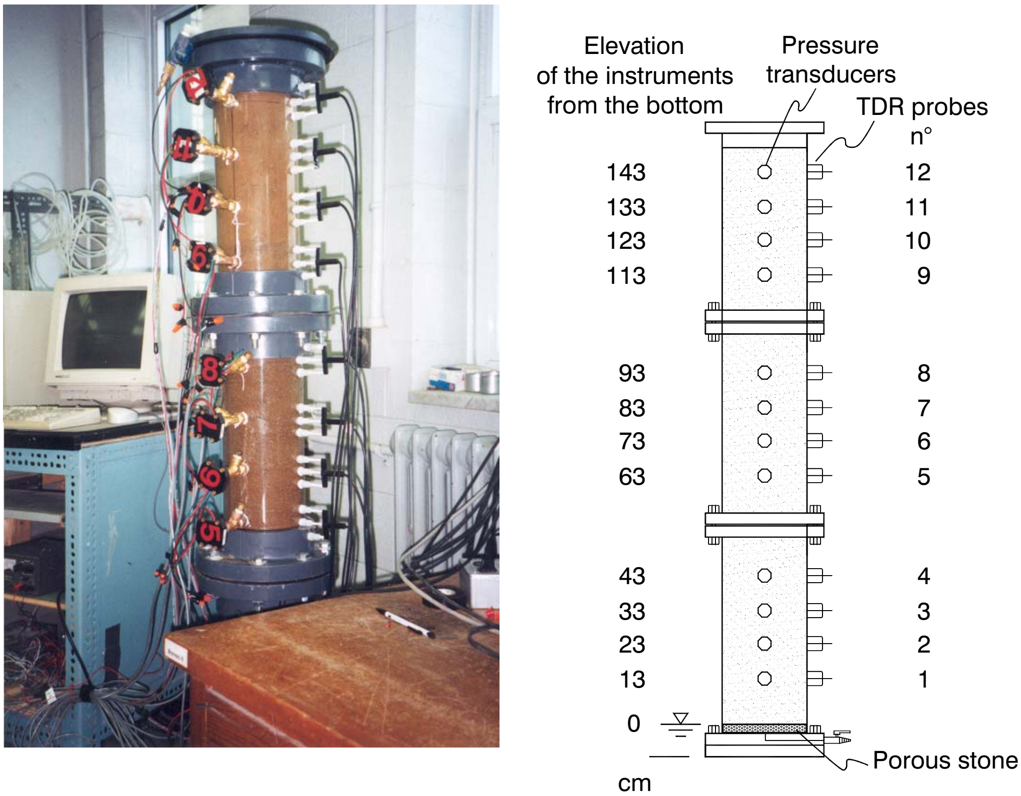

Figure 1. Photo and configuration of the free drainage column test conducted by Bédard (2003)

Using the data $u_w(z,t)$ and $\theta(z,t)$ gathered from the column tests, the unsaturated hydraulic conductivity function k_u was obtained with the transient drainage-flux interpretation method (Hillel 1998, Dirksen 1999). This approach is based on the following general expression:

$$k_u = \frac{\int_0^z \frac{\partial\theta(z,t)}{\partial t}dz}{-\left[\frac{\partial H(z,t)}{\partial z}\right]_z} \qquad [13]$$

where t is time (T), z is the elevation (L), and H is the hydraulic head [L]. The hydraulic head H is the sum of the gravitational head or elevation z (positive above and negative below an arbitrary reference level z = 0) and the pressure head u_w.

The numerator of the right side term is the specific flux [LT^{-1}], while the denominator represents the hydraulic gradient [-] at position z. The reference level z=0 was fixed at the sand surface (near the top of the column). The specific flux at the bottom of a given layer with a thickness Δz, was obtained by applying $[\theta(t_i)-\theta(t_{i+1})/(t_{i+1}- t_i)]\Delta z$, where t_i and t_{i+1} correspond to two consecutive data measurement times. The specific flux at a given depth z is obtained by a summation of the specific flux calculated at the base of all layers overlying that depth. The hydraulic gradient at depth z_j and time t_i is obtained as the slope of the trend line through the plot of the hydraulic heads H(z) at three successive suction measurement positions z_{j-1}, z_j and z_{j+1}.

The water distribution within the sand obtained at equilibrium is compared to the WRC predicted with the MK model (with m =1/C_U and a_c =0.01) in Fig. 2. Predicted and measured values are fairly close for this sand placed in the column with $0.31 \le n \le 0.33$. The AEV of the sand is about 25 cm. Possible hysteresis in the WRC are ignored in this sample application.

The relative hydraulic conductivity function k_r is calculated from equation 8 using the measured k_u and k_s (=5.45×10^{-3} cm/s), and from the Mualem model (eq. 11) with the MAT-k_r-MK code based on the predicted WRC. Calculated results and measured values are presented as $k_r(\psi)$ in Fig. 3. There is a fairly good agreement considering the dispersion of results (due to experimental factors not discussed here); note that readings in the saturated zone (for suctions under about 20 cm) were relatively unstable and have not been used to obtain k_u.

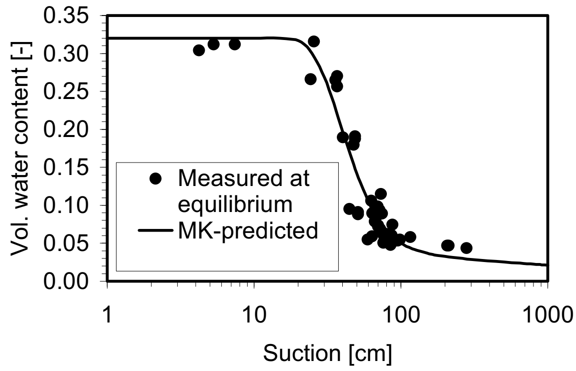

Figure 2. Comparison between measured values and predicted WRC with the MK model at equilibrium for the four drainage tests on the sand.

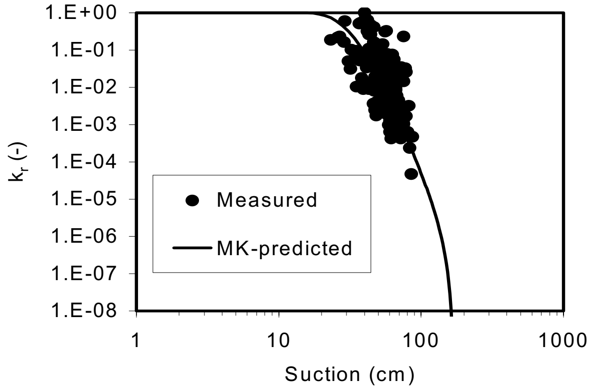

Figure 3. Measured and predicted relative hydraulic conductivity $k_r(\psi)$ for the sand studied by the Authors.

5.2 Data from the literature

The proposed predictive approach was also tested using some published experimental data (see Table 1). The data were taken from Soeiro (1964) and from the UNSODA database (Leij et al. 1996, Nemes et al. 2001). Some results ensuing from a few tests are given here to illustrate the observed tendency.

The predicted (with fixed m and a_c values) and measured values of the WRC for the tests performed on two sands by Soeiro (1964) are shown in Figure 4. These show that the agreement is good for these two sands. The MAT-k_r-MK model was then used to predict the k_r function. As shown in Figure 4 (right side), the agreement is also fairly good.

The results taken from the UNSODA database have been divided into three groups according to the initial porosity of the soils. In the case of dense materials (with n smaller than about 0.33), the prediction of the WRC is usually good (Figure 5). The ensuing k_r function calculated with the MAT-k_r-MK code is also deemed satisfactory in most cases (at least for $\psi \leq \psi_r$).

For moderately loose soils (with $0.33 < n \leq 0.4$ approximately), the prediction obtained for the WRC with the MK model is still satisfactory although not as good as for denser materials. When the predicted WRC is close to the measured values (see Figure 6, left side), the predicted k_r function compares very well with the measured data (Figure 6, right side).

In the case of loose materials (with n larger than about 0.4), the WRC expressed in the usual way (i.e. θ-ψ, assuming constant n during the tests) has a different shape. It shows an apparent decline of the volumetric water content at low suction without a clear AEV (Air Entry Value; see Figure 7, left side). This type of behavior is encountered in compressible materials where suction may induce volumetric changes. The MK equations shown here do not take this phenomenon into account, so the predicted WRC can not match correctly the experimental value. In this case, the k_r function obtained from the MAT-k_r-MK code is quite different from the measured values (see Figure 7; right side).

6. DISCUSSION AND CONCLUSION

The procedure proposed allows the user to predict the relative unsaturated hydraulic conductivity k_r for granular materials, using their basic geotechnical properties. The prediction requires the effective diameter D_{10}, the uniformity coefficient C_U, and the void ratio e. The equations developed have been implemented into MATLAB, to perform the numerical calculations to obtain k_r. The applicability of the proposed approach was shown by comparing experimental data to results predicted using the Mualem solution for k_r. The predictions made with the MK model for the WRC and the k_r function, are shown to be valid as long as the basic assumption of constant porosity remains applicable during the test.

With increasing porosity, it seems that compressibility can affect significantly the WRC measurement. Discrepancies are then observed between predicted and measured values. For such loose and compressible materials, volume changes must be taken into account to properly represent (and predict) the WRC and the ensuing hydraulic conductivity function. A modified version of the MK model has been developed for that purpose, and it will be the topic of an upcoming publication.

On the other hand, various sources of error should also be kept in mind when predictive hydraulic modeling is performed. In the case of k_r (and k_u), there is for instance a significant uncertainty on the hydraulic conductivity close and beyond the residual water content (θ_r) of the soil. Other aspects that need further investigation include the means to introduce hysteresis effects into the model formulations and the actual quality of the measured data (WRC and k_u values). Ongoing work is being performed by the Authors in these areas.

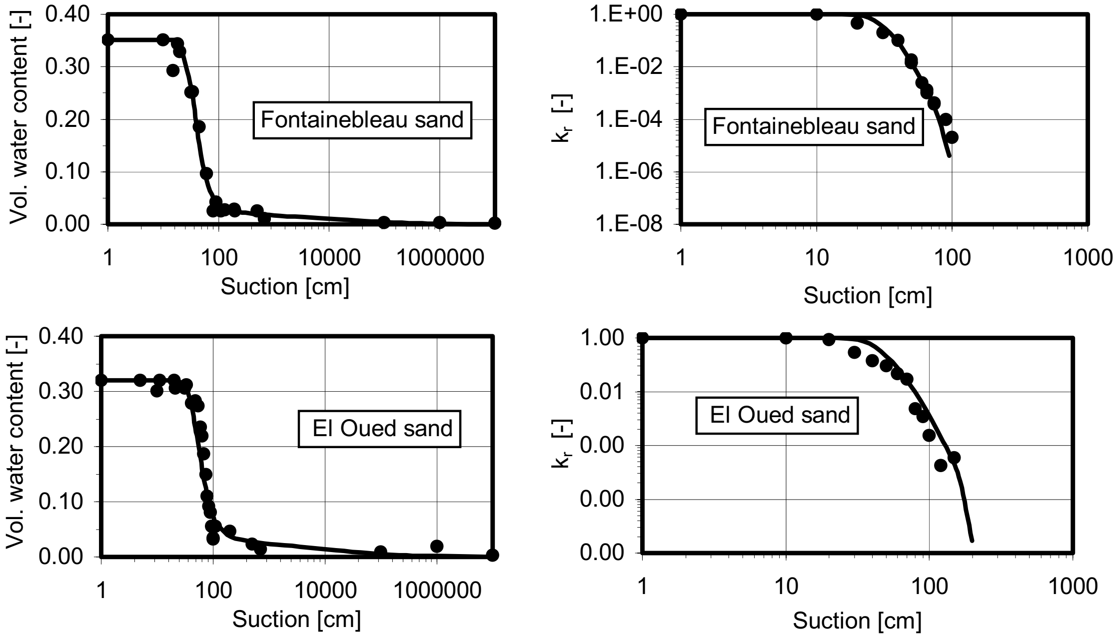

Figure 4. Comparison between measured (dots) and predicted (full line) WRC (left side) and k_r values (right side) for the data taken from Soeiro (1964).

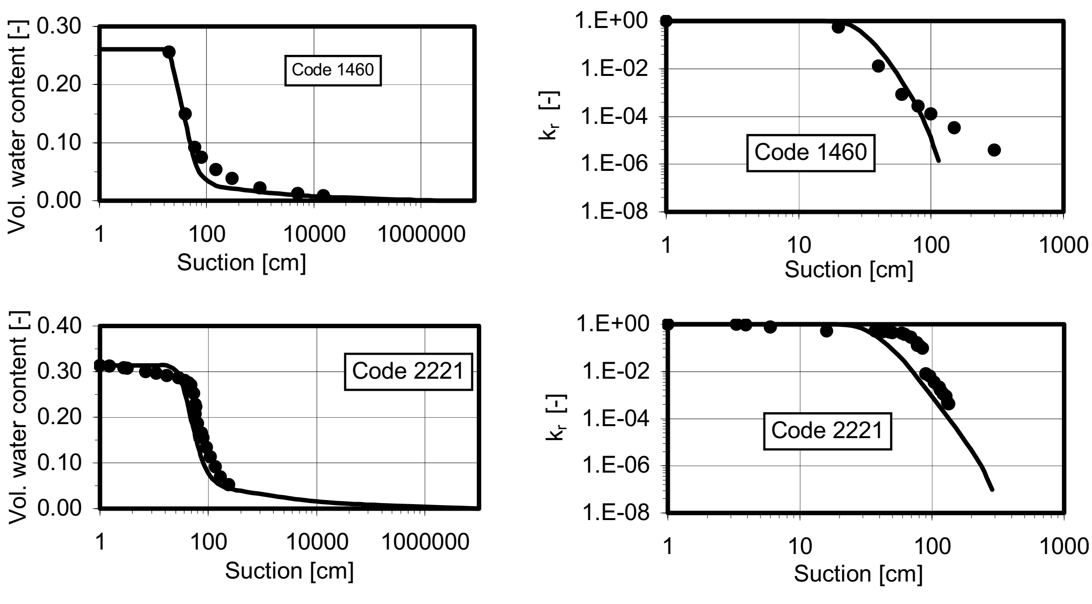

Figure 5. Comparison between measured (dots) and predicted (full line) WRC (left side) and k_r values (right side) for data taken from the UNSODA database, with a porosity $n \leq 0.33$.

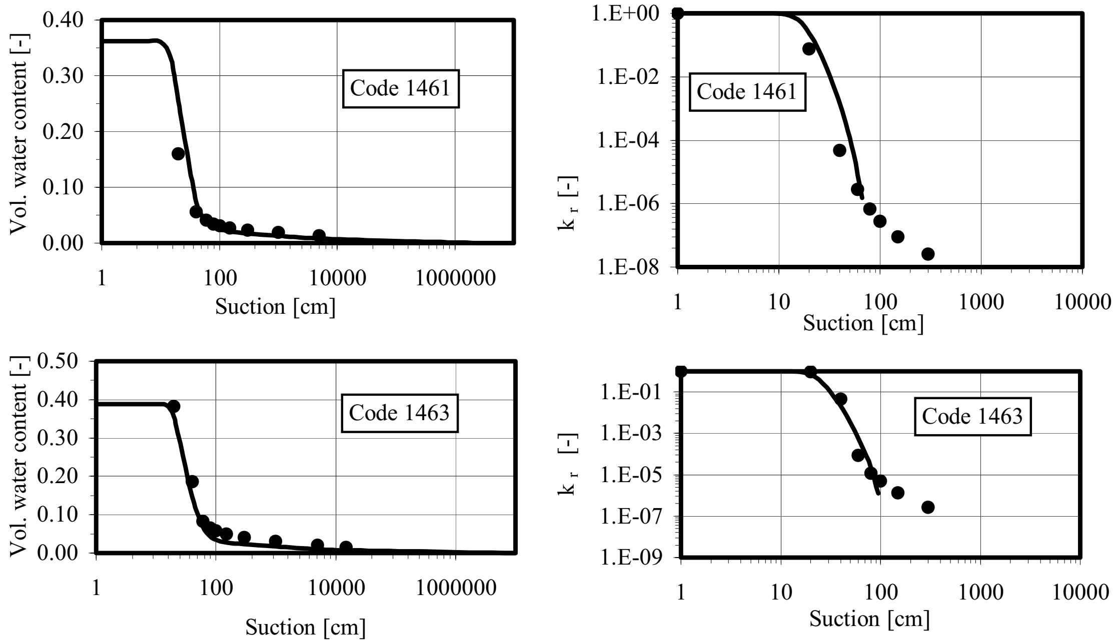

Figure 6. Comparison between measured (dots) and predicted (full line) WRC (left side) and k_r values (right side) for data taken from the UNSODA database, with a porosity 0.33< n ≤ 0.40.

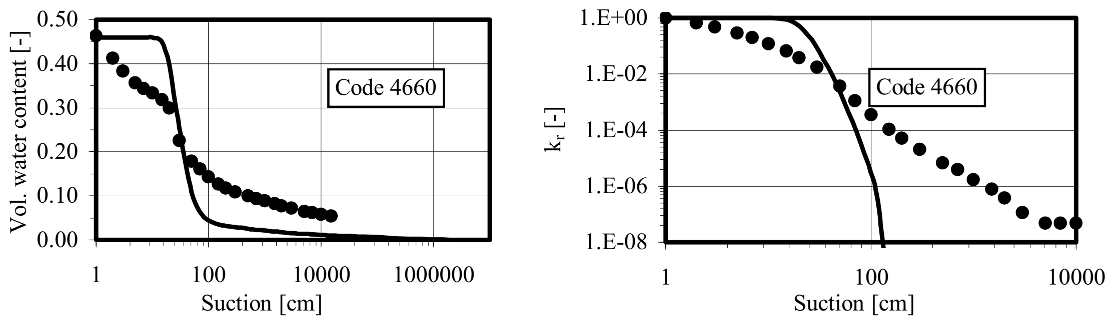

Figure 7. Comparison between measured (dots) and predicted (full line) WRC (left side) and k_r values (right side) for typical data taken from the UNSODA database with a porosity n > 0.40

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