FINITE ELEMENT STOCHASTIC ANALYSIS

Murray Fredlund, Ph.D., P.Eng., SoilVision Systems Ltd., Saskatoon, SK

ABSTRACT
Numerical models can be valuable tools in the prediction of seepage. The results can often be misleading if proper variational analysis is not performed. Measurement of the soil-water characteristic curve (SWCC) or saturated and unsaturated hydraulic conductivity information is often neglected or minimized. The modeller is therefore forced to develop a seepage model based on estimated soil properties. The development of such models based on a best estimate of soil parameters is very often of little value. A stochastic analysis of the results of finite element models is often required in order to properly interpret what information a numerical model is providing.

RÉSUMÉ
Analyse stochastique par éléments finis.

1. INTRODUCTION

The use of numerical models in the regulatory process has increased dramatically in the past few years. The use of such finite element or finite difference models has become accepted practice. There is great benefit in the use of these models as they provide to us a snapshot of the theoretical behavior of a system given very specific constants.

The danger of these models, however, is that they can easily provide us with a false sense of security. In many cases the models do not accommodate the material heterogeneity of a physical system. Consideration is often not provided for experimental error. It is commonplace, for example, for measurements of saturated hydraulic conductivity to vary between ½ to 2 orders of magnitude when measurements are taken on the same soil sample. As results are often sensitive to hydraulic conductivity, this presents a significant modeling problem. It may be stated that a single run of a numerical model given static parameters tells us very little with regards to the behavior of a physical system. This paper examines methods of accommodating stochastic variation in seepage numerical models such that realistic laboratory and field sampling variables may be accommodated.

2. SEEPAGE THEORY

In virtually all studies of flow in the unsaturated zone, the fluid motion is assumed to obey the classical Richards equation (Hillel, 1980;Bear, 1972). This equation may be written in several forms. The three forms of the unsaturated flow equation are identified as the “h-based” form, the “θ-based” form, and the “mixed form”. A transient form of the H-based formulation is presented below.

$$\frac{\partial}{\partial x}\left(kx(\psi)\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(ky(\psi)\frac{\partial h}{\partial y}\right)=m^2{}_w\gamma_w\frac{\partial h}{\partial t}$$

Where:
- h = total head,
- kx = hydraulic conductivity of the soil in the x direction,
- ky = hydraulic conductivity of the soil in the y direction,
- gw = the unit weight of water (9.81 kN/m3),
- m2w = the slope of the soil-water characteristic curve.

The partial differential equation essentially equates flow into and out of a unit volume to the resulting change in storage. The equation appears simple but is plagued by a number of difficulties in obtaining solutions using the finite element or finite difference method. The storage curve and the permeability are both highly non-linear for an unsaturated soil. This causes numerical instability that may be reduced through the application of automatic mesh refinement. The form of the Richards equation presented above is also susceptible to water-balance errors.

There are several advantages to the θ-based form. One advantage is that it can be formulated to be perfectly mass-conservative. It is not commonly used, however, because this form of the Richards equation degenerates in fully saturated media and because material discontinuities produce discontinuous θ profiles.

The h-based form of the Richards equation is the most commonly implemented form. Its primary drawback is that it can suffer from poor mass-balance in transient problems. This problem is exacerbated by problems with a highly non-linear soil-water characteristic curve.

Celia (1990) proposed a “mixed form” of the Richards equation that was designed to improve the mass-balance of the “h-based” formulation.

$$\frac{\partial}{\partial x}\left(kx(\psi)\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(ky(\psi)\frac{\partial h}{\partial y}\right)=\frac{\partial \theta}{\partial t}$$

Where:

h = total head,
kx = hydraulic conductivity of the soil in the x direction,
ky = hydraulic conductivity of the soil in the y direction,
θ = volumetric water content.

Translation of the governing equation to 3D is presented below.

$$\frac{\partial}{\partial x}\left(kx(\psi)\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(ky(\psi)\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(kz(\psi)\frac{\partial h}{\partial z}\right)=\frac{\partial \theta}{\partial t}$$

where:

h = total head,
kx = hydraulic conductivity of the soil in the x direction,
ky = hydraulic conductivity of the soil in the y direction,
kz = hydraulic conductivity of the soil in the z direction,
gw = the unit weight of water (9.81 kN/m3),
θ = volumetric water content.

The issues regarding the solution of the 3D governing partial differential equation are similar to 2D. While the equations appear concise, there are significant numerical pitfalls related to the solution of these equations. The results can vary dramatically dependant upon the soil properties provided.

3. NEED FOR STOCHASTIC ANALYSIS

It was during a first-year laboratory assignment that the importance of variation was impressed on me. We were presented with our electrical system and a formula governing its behaviour. Our assignment was to run the system, collect input data, then use the formula to calculate results. Our laboratory group did precisely as was asked and to our surprise, received a surprisingly low mark. The reason for our low mark, it was discovered, was that we had neglected to incorporate possible limits of variation in our answer, given the possible variation errors in our input parameters. Sixteen years later I have discovered that the state of practice of geotechnical engineering does not apply the concepts presented in first-year engineering when it comes to numerical modelling of soil processes.

Numerical models are routinely set up and run in geotechnical practice with little regard for possible variation in input parameters. A single run of a typical finite element model gives little information, and should not be heavily weighted in the regulatory process. Variational analysis in some form should always be performed to answer the question, “What is the model truly telling us?”

4. EXAMPLE 1: UNSATURATED FLOW IN CLAY DAM

One example of a seepage analysis involves determining the amount of flow through the unsaturated (vadose) zone in an earth dam. A clay core is implemented in this example to dissipate the head accumulated on the upstream side of the dam. The physical dimensions of the dam are presented in Figure 1. The model was setup and run using the SVFlux (Fredlund, 2002) seepage software package.

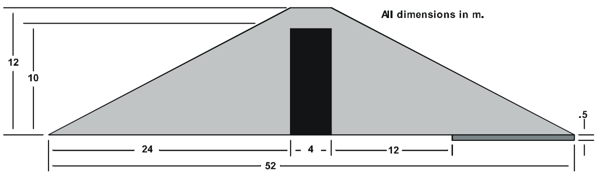

Figure 1 Cross-section of example dam (Stianson, 2004)

The amount of flow proceeding over the clay core and through the unsaturated zone is largely controlled by the unsaturated hydraulic conductivity. The unsaturated hydraulic conductivity is frequently estimated, as laboratory costs may be quite high ($7,000 - $10,000 CAD). In typical estimation methods the slope of the unsaturated hydraulic conductivity curve is related to the slope of the soil-water characteristic curve (SWCC) through a power function.

In this particular example the Modified Campbell (Fredlund, 1996,2004) method of estimating the unsaturated hydraulic conductivity was used in this example. The Modified Campbell method uses the Fredlund and Xing (1994) equation to represent the soil-water characteristic curve as the basis for it’s estimation of unsaturated hydraulic conductivity. The Modified Campbell equation is presented below. The slope of the unsaturated portion of the curve is controlled principally by the “p” parameter.

$$k(\psi)=(k_s-k_{\min})\left[\left[1-\frac{\ln\left(1+\frac{\psi}{h_r}\right)}{\ln\left(1+\frac{10^6}{h_r}\right)}\right]\left[\frac{1}{\left[\ln\left[\exp(1)+\left(\frac{\psi}{a_f}\right)^{n_f}\right]\right]^{m_f}}\right]\right]^{p}+k_{\min}$$

where:

h_r = suction at residual water content,

Ψ	= soil suction,
k_s	= saturated hydraulic conductivity,
k_{min}	= minimum hydraulic conductivity,
a_f	= Fredlund and Xing “a” parameter,
n_f	= Fredlund and Xing “n” parameter,
m_f	= Fredlund and Xing “m” parameter,
p	= Modified Campbell “p” parameter.

A Monte Carlo analysis was then set up that allowed the “p” parameter in the Modified Campbell method to vary such that the mean value was 5 and the standard deviation was 2. 200 variations were generated and the resulting normal distribution is presented in Figure 2. The conductivity of the core material was not modified.

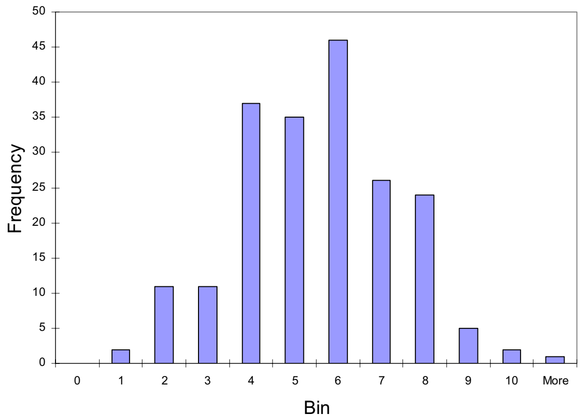

Figure 2 Variation of the "p" parameter in the Modified Campbell equation

A series of 200 runs were used with the Monte Carlo analysis. The results are presented in Figure 3.

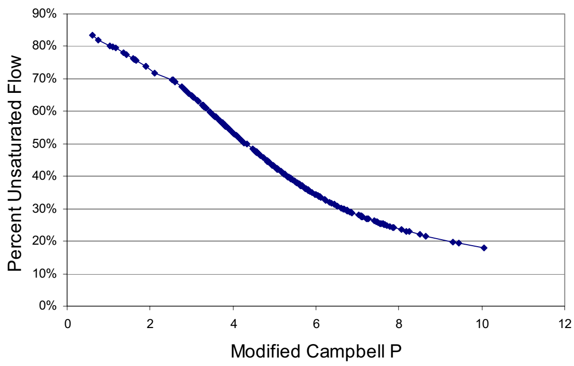

Figure 3 Percent unsaturated flow as related to the slope of the unsaturated hydraulic conductivity curve

The results indicate that the amount of unsaturated flow over the core is highly sensitive to the slope of the unsaturated hydraulic conductivity function. Overall, the amount of flow through the dam as a percentage of total flow varies between 18% and 83%. This example illustrates the importance of a sensitivity analysis when interpreting model results.

It would be easy to run this example using average model parameters. Such model parameters would give the impression that approximately 50% to 60% of flow goes over the core and through the unsaturated zone (using average values of 3.8-5.0 for the “p” parameter of the Modified Campbell equation. Such an impression would be misleading with regards to the results presented by the full stochastic analysis.

It should be noted that the current model was set up using average soil properties. It would also be possible to get variation in results through the variation of other soil parameters such as the air-entry value (AEV) of the soil-water characteristic curve used for the primary dam material.

5. EXAMPLE 2: FOOTING DEFORMATIONS

Calculations involving stresses and deformations beneath a strip footing form an important part of the design process. The deformations caused by the application of load are central to the design tolerances. In a way similar to other finite element models there are certain soil properties that are sensitive and soil properties that are not sensitive. The determination of the sensitive soil properties is of paramount importance. Such a determination can be made through the use of stochastic analysis.

The geometry for the mentioned example problem is shown in Figure 4. A footing is placed at the corner of a 30m x 30m soil region. A load expression of 1 kPa is then applied at the base of the footing and the resulting stresses and deformations are calculated.

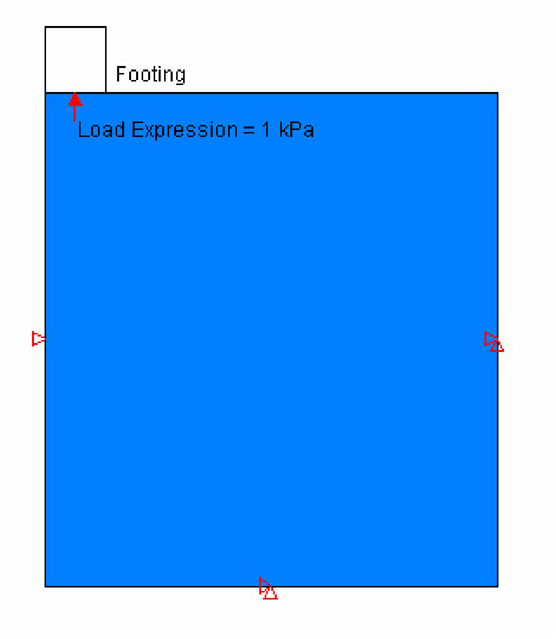

Figure 4 Geometry of strip footing

In this particular analysis the Poisson's ratio was varied via the Monte Carlo method with a mean value of 0.4 and a standard deviation of 0.03. 200 runs of the problem were generated and solved. The distribution of Poisson's Ratio used in this example is shown in Figure 5.

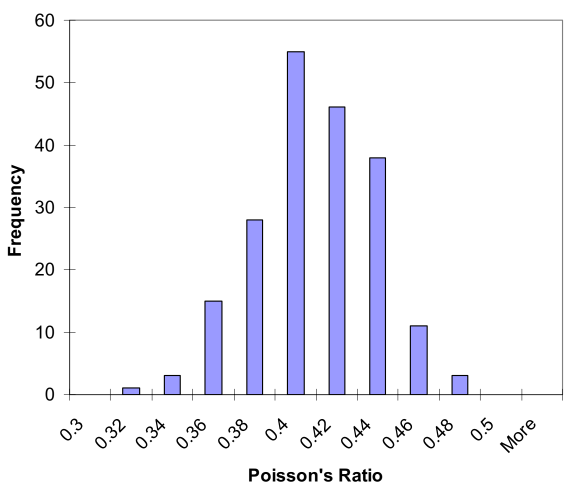

Figure 5 Frequency of Poisson's Ratio used in example

The model results show an insensitivity to stresses in the y direction but a sensitivity to vertical deformations. Deformations were summarized at 3m depth increments beginning at a depth of 1m. The stress deformation beneath the footing is shown in Figure 6. The resulting deformations as a function of model run number are shown in Figure 7. Model runs were organized in terms of increasing Poisson's Ratio values.

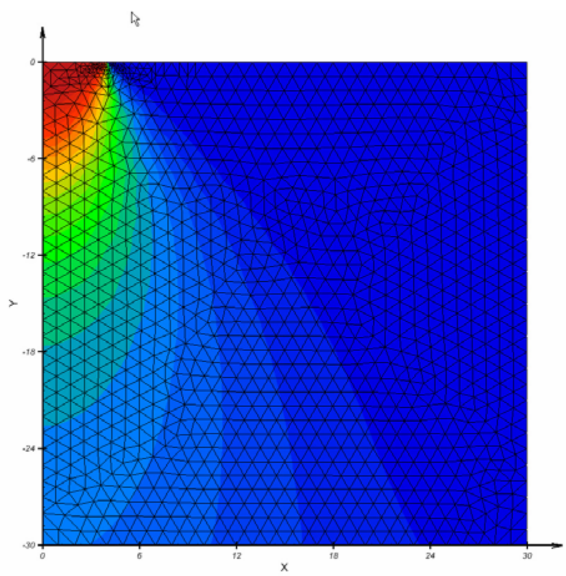

Figure 6 Vertical stress distribution beneath strip footing

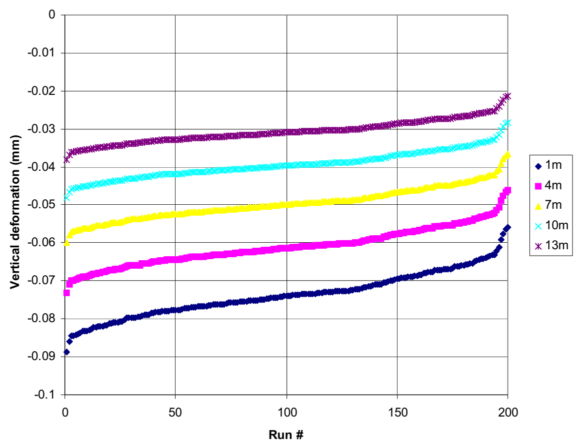

Figure 7 Deformations plotted as a function of model run.

Stochastic analysis gives us a method of determining variability of output. In this example we can include the possible variation of output deformations given the possible variation in input soil properties. The resulting analysis increases our ability to comprehend the value of the numerical results.

6. CONCLUSIONS

Probabilistic methods have not been as widely used in geotechnical engineering as might be expected. Often it is the difficulty in application that results in avoidance. The application of probabilistic methods in finite element analysis has typically not been performed because models lacked i) an ability to batch solve a group of differing problems. This limitation has largely been overcome given the application of automatic mesh generation and automatic mesh refinement. Batches of problems may be set up and run with varying soil properties and the mesh will automatically optimize for each scenario.

A single run of a finite element model is often of little value to the practicing engineer. A group of runs based on a certain varying of soil properties provides much improved information regarding the possible variance of model output. The value of finite element analysis is dramatically improved by implementing the techniques of stochastic analysis. As a result, consultants can provide clients with a statistical basis for their modeling results. The result of the application of this technology will result in increased clarity of regulation guidelines as well as improved defensibility of modeling results by geoconsultants.

“Probabilistic methods, while not a substitute for traditional deterministic design methods, do offer a systematic and quantitative way of accounting for uncertainties encountered by geotechnical engineers, and they are most effective when used to organize and quantify these uncertainties for engineering designs and decisions.” (NRC, 1995)

7. REFERENCES

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