Contemporary approaches to computing failure in geomaterials

Richard G. Wan, Mauricio Pinheiro
Dept. of Civil Engineering, University of Calgary, Calgary, Alberta, Canada
Qiang Li
Golder Associates, Calgary, Alberta, Canada

ABSTRACT
Conventional analysis of failure characterized by unlimited strains under constant stress may not be sufficient since instability phenomena may occur before plastic limit conditions are reached. For example, loose saturated sand, when sheared under undrained axisymmetric loading conditions with stress control, spontaneously collapses after the deviatoric stress reaches a peak value well below the Mohr-Coulomb plastic limit. This mode of failure is diffuse with no apparent failure plane (shear band) as in the case of localized deformations. It corresponds to the violation of Hill's stability condition (second-order work) and the loss of controllability at material level, as recent theoretical studies have shown. This paper briefly discusses the phenomenon of instability and its relevance to practice through the second-order work criterion using both continuum and discrete element frameworks.

RÉSUMÉ
L'analyse conventionnelle où la rupture est définie par des grandes déformations à charge constante est insuffisante car des phénomènes d'instabilité se manifestent avant la condition de limite plastique soit atteinte. Il est connu qu'un sable lâche s'effondre avant la limite de plastique définie par le critère de Mohr-Coulomb, celle sous conditions non-drainées au pic des contraintes déviatoires en pilotant en contraintes. Ce mode de rupture est diffus sans localisation de déformations. Théoriquement, cette condition correspond à l'annulation du travail du second ordre selon le critère de d'instabilité matérielle de Hill et la perte de contrôlabilité du chargement. Ainsi, cette communication aborde la problématique d'instabilité et sa pertinence en pratique par le critère du travail du second ordre dans une approche continue ainsi que discrète.

1 INTRODUCTION

Geomaterials belong to the class of materials that include soil, rock and concrete where frictional cohesive strength is manifested as a function of the size of the particles and the degree of bonding between them. It is the interaction of the solid skeleton with a fluid phase that makes every geomechanics/geotechnical problem so challenging to analyze. Furthermore, the microstructural aspects of geomaterials give rise to a very rich and complex behaviour that needs to be described into any constitutive model. Traditionally, the behaviour of geomaterials has been described using the theory of elasticity to arrive at useful closed-form solutions. Later, more elaborate approaches based on the theory of plasticity along with limit analysis have led to important milestones in soil mechanics whereby the notion of material failure can be described by computing collapse loads. Examples of this pertain to the bearing capacity formula in foundations, the evaluation of lateral earth pressures in excavations and the generalized limit equilibrium analysis for studying slope stability. On the other hand, the advent of numerical techniques such as the finite element method has facilitated the analysis of geostructures under complex loading situations in a boundary value setting by using sophisticated constitutive models to describe soil behaviour with high fidelity.

There are still many geotechnical problems that cannot be solved using traditional methods of analysis even though the underlying concepts are largely well accepted and have a strong basis. An example is the failure of low angled submarine slopes despite their inclination being well below the friction angle. The sudden collapse of a slope without any substantial change in loading conditions cannot be explained within simple theories such as the traditional limit equilibrium analysis so pervasively used in geotechnical engineering.

The central issue hinges around the question of failure in geomaterials which is largely a characteristic of their constitutive behaviour. It is highly desirable to model complex material behaviours that originate from non-linearity, stress-strain history dependencies, and development of anisotropy. However, as the richness of the material description increases, the analysis becomes more involved and non-uniqueness, bifurcation, as well as material instability emerge as subtle distinctive features of failure. As such, it is not sufficient to solely describe failure as a plastic limit condition under which stresses reach a limiting value, but also the notion of material instability must be examined. Other modes of failure such as diffuse instability can appear well before reaching the plastic limit surface, normally represented by a Mohr-Coulomb surface. This is next discussed and a diffuse bifurcation criterion is evoked in which the second-order work is calculated and its sign examined.

2 SECOND-ORDER WORK

Darve et al. (2004) evoked the paradoxical problem where failure takes place inside the plastic limit surface under particular loading conditions such as in undrained stress-controlled tests on loose sand. The authors, from theoretical studies, have demonstrated that this mode of failure does not generally involve any localized failure plane but rather diffuse deformations, whereby the material collapses in a catastrophic manner releasing kinetic energy without any need for further incremental energy input to the system. Such a condition can be mathematically interpreted from energy arguments leading to a criterion based on the sign of the second-order work (Hill, 1958), i.e. $d^2W = d\boldsymbol{\sigma}{:}d\boldsymbol{\varepsilon}$. Basically, starting from an equilibrium state, we seek various directions of perturbations in the velocity field for any unstable states that may drift from equilibrium and thereby allow a spontaneous change of potential energy into kinetic energy. This criterion can be employed to capture the existence of material instability well inside, with stress states far away from, the classical plastic limit surface.

For frictional materials which exhibit non-associated plastic flow, one may encounter stress paths for which the positivity of the second-order work may not be fulfilled, even before the failure surface has been reached. Notably, materials with tendencies to compact (densify) during shearing can become unstable within the failure surface through static liquefaction in undrained conditions (Lo and Chu, 1991). Chu and Leong (2001) from experiments on both dense and loose sand concluded that material instability can be regarded as a special kind of strain softening, which is again loading condition dependent (either stress control or strain control). Here, strain softening is a condition when density decreases with increasing strain. It is noted that instability is not synonymous with failure for granular materials, as the failure surface does not pass through each of the points at which instability was produced.

Diffuse instability is signalled by the non-positiveness of the second order work, i.e. $d^2W = d\boldsymbol{\sigma}{:}d\boldsymbol{\varepsilon} \leq 0$. Therefore, it largely hinges on the constitutive relationship $d\boldsymbol{\sigma} = \mathbf{D}{:}d\boldsymbol{\varepsilon}$ under investigation and leads to the condition:

$$d^2W \equiv d\boldsymbol{\sigma} : d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^T : \mathbf{D} : d\boldsymbol{\varepsilon} \leq 0 \qquad [1]$$

From Eq. [1], a negative second order work signifies that the local constitutive matrix loses positive-definiteness. In the context of solving a boundary value problem via finite element computations, the associated equations lose ellipticity (become non-quadratic), and hence both uniqueness and well-posedness of the solution are not guaranteed. Thus, the second order work criterion must be checked besides the plastic limit criterion in any finite element analysis to detect diffuse modes of failure inside the plastic limit surface.

3 STABILITY ANALYSIS

In the following sections, we apply Hill's second-order work criterion to analyze the stability phenomenon from various perspectives. First, we present our own results from an experimental viewpoint, where two-dimensional (2-D) photoelastic prismatic particles are used as to emulate granular material behaviour. Then, we move on to numerical approaches, where stability is investigated at both mesoscopic (constitutive) and macroscopic (boundary value problems) levels. The finite element method and the discrete element method are used.

3.1 Experimental Approach: Photoelastic Material

The mechanical behaviour of an assembly of about 200 photoelastic polyurethane particles is the main subject of analysis here. This analogue granular material, composed of an ensemble of pentagonal photoelastic particles, was tested in a biaxial apparatus as shown in Figure 1. In this apparatus, four high precision actuators are used to load the approximately 100 mm × 100 mm specimen, under stress and strain control modes (Al-Mamun, 2004). One scenario is to apply axial and lateral forces F_1 and F_2 to the specimen and thereafter measure the resulting displacements D_1 and D_2. A series of tests was conducted along proportional strain paths so that the specimen response in the complete compression and dilation regimes could be explored. It is important to highlight that the stress-strain response of a soil element is generally highly path dependent. Hence it is essential to study the stability of granular materials along a wide range of paths. Particularly, the control of strain path is a better alternative to the control of stress path since in the latter situation a sample may run into an inadmissible state at the onset of instability. In fact, the strain path control test, in which either compaction or dilation is imposed to the specimen, generalizes the actual drainage condition of soils in the field where undrained condition is only a special case. It gives an opportunity to investigate pre-failure flow (such as instability, softening etc.) of granular soils under more general conditions through a series of strain paths. Dilation has immense influence on the overall response of granular materials. Through conventional triaxial testing (either drained or undrained), we cannot control the lateral extension of the specimen. Instead, by controlling the lateral strain a much detailed understanding of the specimen dilation and subsequent response can be studied. The stability of a specimen for any combination of compression and extension can be studied.

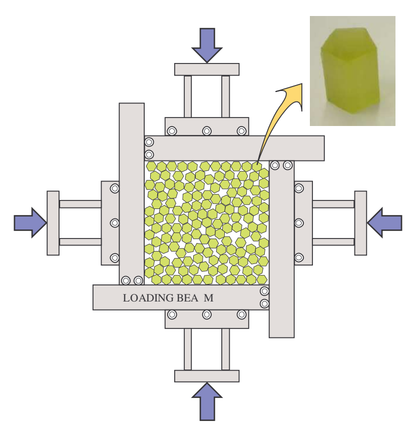

Figure 1: Schematic view of the biaxial apparatus. Zoom at a photoelastic polyurethane particle shown for illustration purpose (modified after Al-Mamun, 2004)

In the following, a stability analysis much like to that one undertaken by Darve and Laouafa (2002) on sands under axisymmetric conditions was carried out here under bi-axial conditions. For that, the second-order work expressed earlier in Eq. (1) is first rearranged as below in terms of the ratio $R = -\Delta D_1/\Delta D_2$ which controls the rate of compression or dilatation:

$$d^2W \equiv \Delta\left(F_1 - F_2/R\right)\Delta D_1 \quad [2]$$

where ΔF_1 (resp. ΔD_1) and ΔF_2 (resp. ΔD_2) are the forces (resp. displacements) acting along axial and lateral axes, respectively. A zero second-order work is thus signalled by a zero slope (at peak point) of the $(F_1 - F_2/R)$ versus D_1 plot. As such, the slope $S = \Delta(F_1 - F_2/R)/\Delta D_1$ is tracked all along during the test which follows a proportional strain path since the value of R is kept constant throughout.

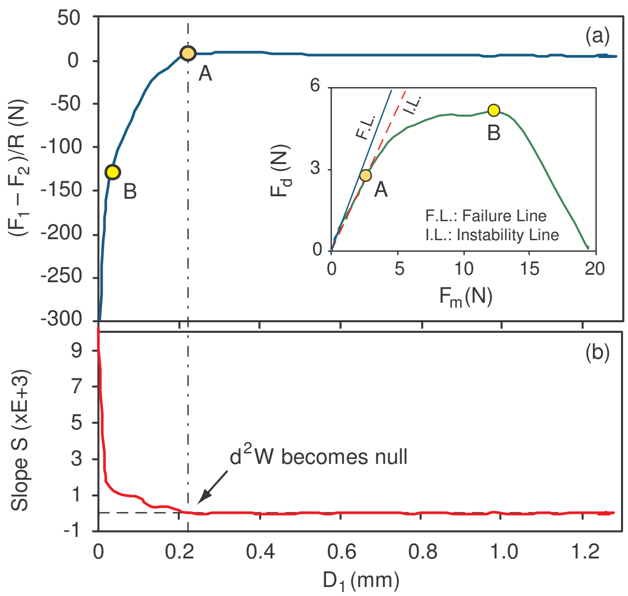

Figure 2: Stability plot for R = 0.05 and 20 N confinement using photoelastic rods (modified after Al-Mamun, 2004)

The stability curve for R = 0.05 which corresponds to force dilatation is shown in Figure 2a and the variation of slope S against axial displacement is plotted in Figure 2b. In the latter plot, the condition of zero slope (S = 0) is clearly shown, thereby indicating material instability. The corresponding force response path is shown in the inset of Figure 2a. From the location of instability point A in the force response path, it is revealed that, the specimen was stable even in the post peak region, see point B in Figure 2a. That is, instability does not occur until the response path reaches the instability line though its slope is negative before that. Furthermore, it is important to point out that point A does not fall on the limit failure line, but on a so-called instability line which lags behind it. This mode of failure which is diffuse will be further discussed within the context of an elasto-plastic constitutive model in section 3.3.1.

3.2 Discrete Approach: DEM Application

The discrete element approach is a numerical technique that regards its elements as discrete, distinct, and interactive, just like sandy soil particles are. This approach was conceived by Cundall (1971) and firstly applied to rock mechanics. Nowadays, almost 40 years later, a few methods with similar philosophy to that originally proposed by Cundall have been formulated. Here we present the published results of stability analysis using the Contact Dynamics Method (CDM). In the contact dynamics theory developed by Moreau (1994), the grains are assumed perfectly rigid and their interactions are described by a simple Coulombian friction law, where no springs, slides, and dashpots are involved. Its numerical algorithm for the time integration is implicit. In DEM developed by Cundall and Strack (1979), contacts between particles are soft; thus penetration between bodies is allowed. The numerical algorithm for the time integration is solved explicitly. Therefore, CDM is essentially different from the DEM (Darve et al. 2004). In many problems that involve large deformations such as flow slides, particle deformations are not important and it is the particle kinematics that governs the whole deformation process. Hence, in this case, the CDM method is more appropriate than the DEM method. Also, the issue of artificial damping and objectivity of computations often encountered in DEM is avoided.

3.2.1 Stability analysis using contact dynamics approach

Here, we illustrate the relevance of the second-order work and its correlation with the burst of kinetic energy in a geotechnical application such as an avalanche.

The simulations were carried out by Darve et al. (2004) using CDM. Figure 3a shows a two-dimensional slope whose top boundary is moved at a constant vertical displacement rate in order to induce failure. The slope is made up of 377 polydispersed circular particles. More details about this simulation can be found in Darve et al. (2004).

Figure 3b shows the displacement fields after the top boundary of the slope has been moved 25.2 mm down. A deeply seated failure mechanism exhibiting a rotational sliding is captured (Darve and Laouafa, 2001). Our greatest interest here, though, is to verify the correspondence between non-positive values of second-order work and bursts of kinetic energy. This is illustrated in Figures 3c, d, where the size of the square symbols is proportional to the magnitude of the variables in analysis. From these figures, the spatial correlation that appears between loci of non-positive second-order works d^2W and bursts of kinetic energy E_c is quite evident. According to Darve et al. (2004), this correlation is not verified grain by grain (some grains with $d^2W > 0$ move while others with $d^2W < 0$ are not moving). Thus, the relation between the local second order work at the grain level and that at the global level remains to be explored. Nonetheless, the main idea about using second-order work as a means to identify instability is promising.

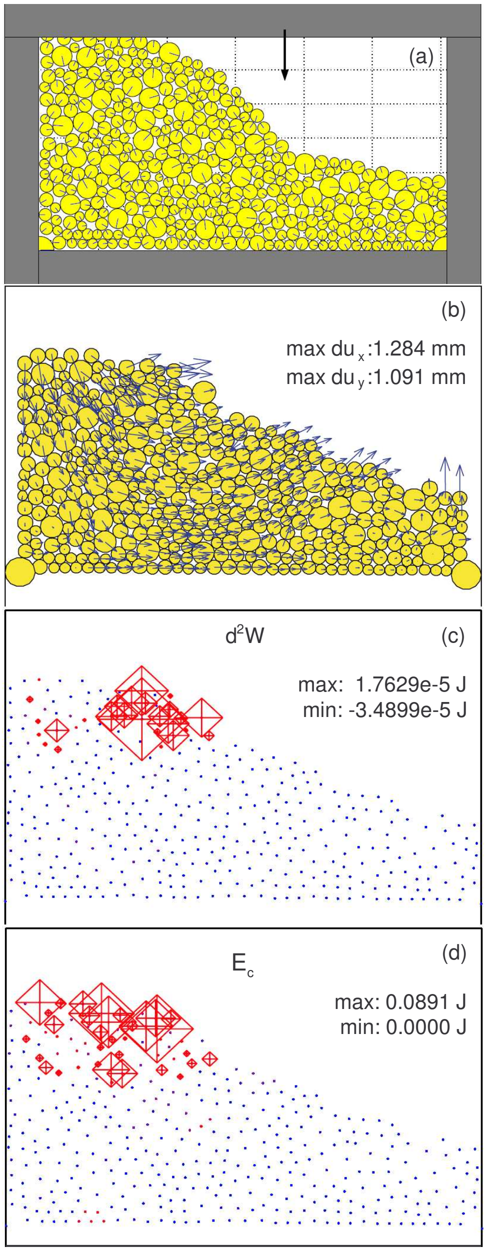

Figure 3: Second-order work in avalanche phenomenon using a 2D contact dynamics code (modified after Darve et al., 2004)

3.3 Continuum Approach: Constitutive Modelling and FEM Applications

Different from the previous section where the discrete approach was taken into account, here we perform our analysis of failure in geomaterials based on principles of continuum mechanics. As such, constitutive models founded on the theory of plasticity are used. We start off the study from a material (element) level (or Gauss point level in finite element setting), and then we extend it to a boundary value problem application.

3.3.1 Stability analysis at local level using an advanced continuum mechanics model

From Eq. (1), we can see that diffuse instability largely hinges on the constitutive behaviour of the geomaterial. As the intricacy and richness of the constitutive model increases so as to reproduce salient features of the geomaterial mechanical behaviour such as non-linearity, strain-softening, dilatancy, and fabric, the sensitivities of the analysis to instabilities increase accordingly. If simple elastic behaviour were considered, diffuse instabilities would never be captured. As such, in this section we use an advanced constitutive model that has embedded into its formulations the effects of pyknotropy (density), barotropy (stress level), and anisotropy (fabric), see Wan and Guo (2004). This model is based on the theory of plasticity and also employs extended concepts of Rowe's stress-dilatancy and critical state theories.

In the analysis below, we carried out a series of stress probes and computed the resulting strain response in each case. The tests presented are confined to axisymmetric conditions and the magnitude of the stress probes is 0.01 kPa.

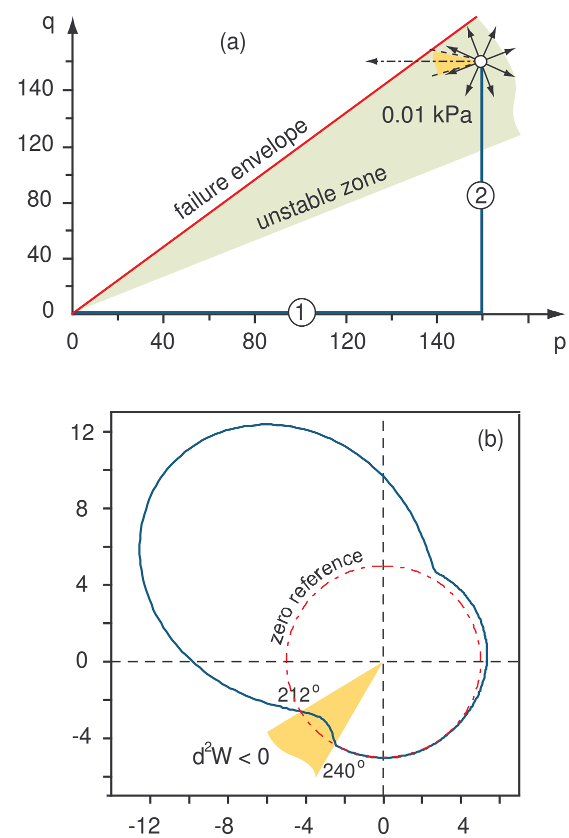

Figure 4: Continuum approach to calculate the second-order work at element level

First, the specimen was brought to an isotropic stress state of 150 kPa, and then it was sheared at constant mean effective stress until the deviatoric stress, q, reached 150 kPa. At that state, which lies inside the zone of instability, the stress probing was conducted as depicted in Figure 4a. It is noted that this unstable zone is determined after defining the lower boundary of stress states for which a negative sign of the second order work is first encountered.

Figure 4b presents the envelope of second-order work values for the stress probe test. In this plot, we observe an unstable zone between probe angles 212º and 240º where $d^2W < 0$. That zone has an important meaning within Hill's stability theory. If a material undergoes a loading process such that the stress path travels through such a zone of instability, the (local) material incremental response will be unstable. In Figure 4a, it is indeed confirmed that the q constant path in drained conditions as encountered in in-situ fluid injection in soils refers to an unstable loading path. In a boundary value setting, depending on the neighbouring local responses, the material may partially fail or even collapse as a whole. That will be more evident in the next section where an excavation under plane strain conditions is simulated.

3.3.2 Stability analysis of a boundary value problem using FEM

In this last section, material failure as a diffuse instability phenomenon is investigated from a macroscopic point of view. In other words, a boundary value problem (BVP) is analyzed and its mechanical response is related to Hill's second-order work criterion. This is a large departure from traditional analyses of slope instability in finite elements with elasto-plasticity. Here, we focus on the aspect of instability being possible for states of stresses within the limit plasticity surface.

As an example of a BVP in geomechanics we present the case study reported by Khoa (2005): a 30 m deep slope excavation under plane strain conditions. The FEM-based commercial software PLAXIS was used in the analysis. For details on PLAXIS, please refer to Brinkgreve (2002). Our main focus in this paper is to examine the nature of material failure and various modes.

The soil mass under investigation is constituted of Hostun dense sand. To properly describe the mechanical behaviour of Hostun sand, the PLAXIS built-in Hardening Soil Model (HSM) was used. HSM shares some similarities with the Wan-Guo (2004) model introduced in Section 3.3.1 as both are isotropic hardening elastoplastic models based on Rowe's stress-dilatancy theory and the critical state concept. In HSM, though, the ultimate failure is captured by Mohr-Coulomb criterion, while in Wan-Guo model a smoothed version of Mohr-Coulomb criterion is used.

The excavation advanced in thirty phases of 1-m deep each. At the end of each phase, isovalues of the second-order work are calculated and interpolated from every Gauss point. In Figure 5b, these isovalues for three different depths (10, 20, and 30 m) are plotted. Stress points that are in the hardening regime (lighter points) or have reached the plastic limit on the Mohr-Coulomb failure envelope (darker points) are indicated in Figure 5a. It is seen that the regions of stress states at plastic limit are generally confined to the surface of the slope with a modest extension into the slope when 30 m of excavation have been completed. Underlying these plastic regions are extensive zones of plastic hardening. It is of interest to compute the second-order work for each excavation step in order to detect any diffuse mode of failure. Figure 5b clearly shows regions in the hardening stress regime that are at zero or negative second order work, and hence potentially unstable from Hill's sense. The zone of negative second-order work evidently identifies the part of the slope where the soil may just collapse and flow in a diffuse manner, despite that the stresses are within the plastic limit surface.

4 CONCLUSIONS

The failure of geomaterials due to unstable phenomena with a diffuse deformation field was the main discussion in this paper. The main argument is that failure is expressed by various subtle modes which occur well before the plastic limit. There is a domain of the stress space below the plastic limit surface where soil behaviour bifurcates and becomes unstable. More precisely, for a certain direction of loading, the material response may not be unique, and unstable modes appear with ensuing increase of kinetic energy. A celebrated case is the collapse behaviour of undrained loose sand under stress controlled loading, which was otherwise stable if strains were controlled. This diffuse type of failure is signalled by the nullity or negativity of the second order work as introduced by Hill (1958). The linkage between the second order work and diffuse failure is demonstrated in three types of analyses: (1) experimental laboratory tests with two-dimensional photoelastic polyurethane particles, (2) discrete element modelling and (3) elasto-plasticity at both material point and boundary value settings. It is concluded that one has to be careful in analyzing the failure of a geostructure (e.g. slope or embankment). The classic approach involving a plastic limit through a failure criterion may not be sufficient, especially cases where flow type of deformations (diffuse) are involved. This mode appears well before localization into a failure plane and must be checked in the analysis by computing the second order work. Thus, any contemporary analysis in geotechnical engineering should consider the possibility of bifurcation of the material response. This is necessarily true as, in view of capturing most salient features of geomaterials, constitutive laws used in such analyses become richer and thus more conducive to bifurcation phenomena. The issue of controllability of material response is raised and it is the loss of control of material behaviour that leads to collapse and ultimate failure. In a boundary value problem setting, a material point is subjected to mixed type of loading (displacement, stress and pore pressure) due to partial drainage and contrasts in the permeability field. A numerical paradigm that accommodates for such mixed loading needs to be developed as most finite element analyses are displacement/strain driven. The question of non-uniqueness of solution must also be examined as in many cases it may just be a natural feature of the engineering problem at hand. A flow slide triggered by some subtle changes in loading conditions (control parameters) is an example of bifurcation in material response with the emergence of diffuse mode and release of kinetic energy without any incremental supply of energy to the system.

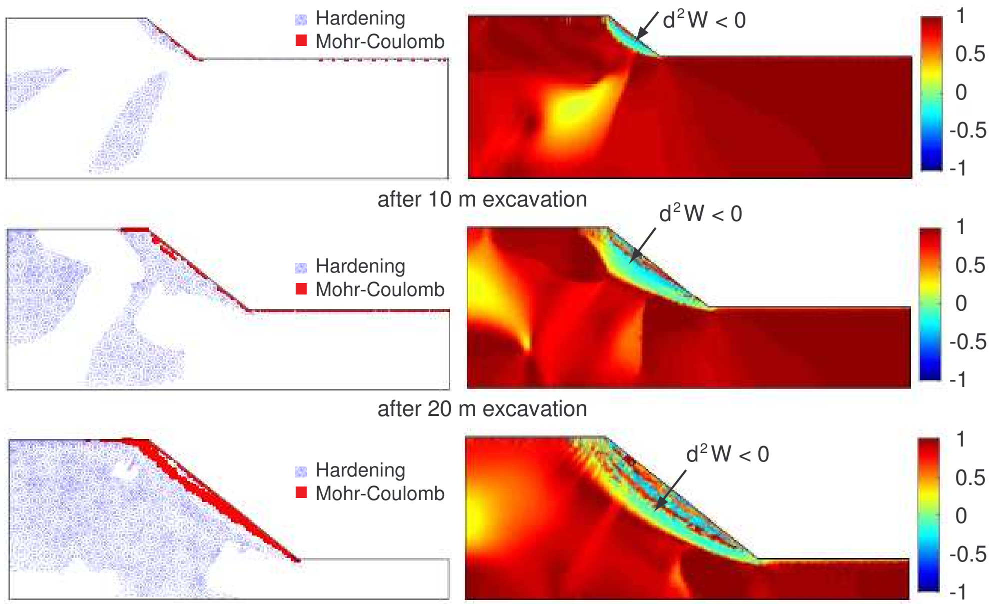

Figure 5: Application to boundary value problems using FEM. Second-order work analysis (Khoa, 2005)

ACKNOWLEDGEMENTS

The financial support provided by the Natural Science and Engineering Research Council (NSERC) is acknowledged.

REFERENCES

Al-Mamun, M.A.M. 2004. *Experimental Study of Granular Material Behavior and Microstructure in a Photoelastic Disk Assembly*. MSc Thesis, U of Calgary, Canada.

Brinkgreve, R.B.J. 2002 *PLAXIS Version 8: Finite Element Code for Soil & Rock Analyses*. Balkema Rotterdam.

Chu, J. and Leong, W.K. 2001. Prefailure strain softening and prefailure instability of sand: A comparative study. *Geotechnique* 51(4): 311-321.

Cundall, P.A. 1971. A Computer Model for Simulating Progressive Large-Scale Movements in Blocky Rock Mechanics. In: *Proc. Symposium Intl Soc. Rock Mech.* Nancy, France, 2: 129-136.

Cundall, P.A. and Strack, O.D.L 1979. A Discrete Num. Model for Granular Assemblies, *Geotechnique* 29(1): 47-65.

Darve, F. and Laouafa, F. 2001. Modelling of granular avalanches as material instabilities. In: Muhlhaus et al. Eds, Zwets and Zeitlinger Publ., *Bifurcation and Localization in Geomechanics*, 29-36.

Darve, F., Servant, G., Laouafa, F., and Khoa, H.D.V. 2004. Failure in geomaterials: continuous and discrete analyses. *Comp. Methods Applied Mech. and Eng.*, 193(27-29): 3057-3085.

Hill, R. 1958. A general theory of uniqueness and stability in elastic-plastic solids. *J. Mech. Phys. Solids*, 6, 236-249.

Itasca, C.G. 1999. *PFC 3D: User Manual*. Itasca Consult. Group, Minneapolis.

Khoa, H.D.V. 2005. *Modélisations des Glissements de Terrain Comme un Problème de Bifurcation*. PhD Thesis, Institut National Polytechnique de Grenoble, France, 214p.

Moreau, J.J. 1994. Some Numerical Methods in Multibody Dynamics, Application to Granular Materials, *Eur. J. Mech. A*, 13(4): 93-114.

Lo, S.-C.R. and Chu, J. 1991. The measurement of K_0 by triaxial strain path testing. *Soils and Foundations*, 31, 181-187.

Nova, R. 1994. Controllability of the incremental response of soil specimens subjected to arbitrary loading programs. *J. Mechanical Behavior of Materials*, 5(2): 193-201.

Rudnicki, J.W. and Rice, J.R., 1975. Conditions for the localization of deformation in pressure sensitive dilatant material. *J. Mech. Phys. Solids*, 23, 371-394.

Wan, R.G. and Guo, P. 2004. Stress-Dilatancy and Fabric Dependencies on Sand Behavior. *J. Eng. Mechanics.* 130(6): 635-645.