Consolidation in multi-layered soils: a hybrid computation scheme

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ABSTRACT
A new approach for computing one-dimensional consolidation in layered soils is presented. Traditionally, Terzaghi's one-dimensional consolidation equation is solved using an explicit finite difference scheme. The explicit approach requires that the timestep remains below some threshold that depends on layer thickness and permeability. For scenarios with large permeability contrasts between layers, a large number of timesteps may be needed to reach a consolidated state. The explicit method therefore becomes impracticably slow as the permeability contrast between layers increases. To overcome this, an implicit approach may be used in which the timestep can be much larger. The implicit solution is always stable, however for very short timesteps, the solution may be inaccurate. A new method is presented here that uses the traditional explicit approach for short times, and then adaptively switches to an implicit solution to compute results at long times. This hybrid strategy significantly speeds up the required calculation time without sacrificing accuracy. Examples are presented showing the accuracy and speed of this new technique.

RÉSUMÉ
Une nouvelle approche pour le calcul unidimensionnel consolidation dans les sols stratifiés est présenté. Traditionnellement, Terzaghi's one-dimensional consolidation équation est résolu en utilisant un schéma explicite aux différences finies. L'approche exige explicitement que le reste de temps au-dessous de certains seuils qui dépend de la perméabilité et l'épaisseur de la couche. Pour les scénarios avec de grands contrastes de perméabilité entre les couches, des millions de timesteps peut être nécessaire pour atteindre un état consolidé. La méthode explicite devient donc impracticably lent que le contraste entre la perméabilité des couches augmente. Pour surmonter ce problème, une approche implicite peut être utilisé dans lequel le timestep peut être beaucoup plus important. La solution est implicite toujours stables, mais pour de très courtes timesteps, la solution peuvent être inexacts. Une nouvelle méthode est présentée ici qui utilise l'approche traditionnelle explicite pour les courts moments, et puis adaptative passe à une solution implicite de calculer les résultats à long temps. Cette stratégie hybride accélère de manière significative le temps de calcul nécessaire, sans sacrifier la précision. Des exemples sont présentés montrant la précision et la rapidité de cette nouvelle technique.

1 INTRODUCTION

When a load is applied to the ground surface (e.g. placement of fill or an embankment) some proportion of the stress is carried by the pore water, producing an excess pore water pressure. As this pressure gradually dissipates, the load is transferred to the soil and downward displacement (consolidation) occurs. For low-permeability clays, this process may take many years and therefore simulation of this phenomenon is important in many engineering applications.

The simplest and most common way to calculate consolidation in practice is to use the one-dimensional consolidation equation developed by Karl Terzaghi in the 1920s. The Terzaghi equation is a form of diffusion equation (like 1D heat flow) and can be solved mathematically for a homogeneous soil with drained or undrained boundary conditions. For a layered soil, the solution can be approximated using finite difference techniques in which different properties are assigned to different layers.

Many software products use an *explicit* finite difference calculation technique. With this approach, the solution space is discretized in space and time. The solution marches forward in time and assumes that information cannot propagate further than the adjacent spatial node over any one time step. For this reason, time steps must be small to maintain stability. The solution works well for near homogeneous materials and short timescales, but the computation time becomes prohibitive as the contrast between layers increases and timescales become long.

It is also possible to solve the problem *implicitly*, such that the excess pore pressures at all nodes are solved simultaneously at the desired time using matrix inversion (Abid and Pyrah, 1988). Since the problem can be solved at any time, there is no need to use small time steps and therefore the solution is much faster for long timescales. However with this approach there is a minimum timestep required to ensure accuracy.

In general, it is possible to use the implicit approach exclusively, but if results are required at early times, it is necessary to run an explicit analysis until the minimum implicit timestep is reached.

In this paper, it is shown how the explicit and implicit techniques can be used together to simulate 1D consolidation for layered soils. At early times, the explicit approach gives accurate results. These results are then used as the initial conditions for the implicit solution, which is used to compute consolidation at later times. Using these techniques in tandem overcomes the drawbacks of each method and ensures speedy and accurate results.

2 FINITE DIFFERENCE METHOD

Vertical consolidation is dictated by Terzhagi's 1D consolidation equation:

$$\frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial z^2} \qquad 1$$

Where u is the excess pore pressure, c_v is the coefficient of consolidation and z is the vertical distance below the ground surface. The excess pressure at any time is calculated from this equation and is then used to calculate the effective stress. Strains can then be calculated depending on the material type and the strains are used to calculate settlement.

This equation can be solved analytically for a single layer. For multiple layers with differing coefficients of consolidation and different thicknesses, a finite difference approach can be used to solve for the pore pressure.

If the problem domain is discretized in time and space as shown in Figure 1, equation 1 can be replaced by:

$$\frac{\partial u}{\partial t} = \frac{u_{i,t+\Delta t} - u_{i,t}}{\Delta t}$$

$$c_v \frac{\partial^2 u}{\partial x^2} = (1-\phi)\left\{\frac{c_v}{(\Delta h)^2}\left[u_{i-1,t} - 2u_{i,t} + u_{i+1,t}\right]\right\} + \phi\left\{\frac{c_v}{(\Delta h)^2}\left[u_{i-1,t+\Delta t} - 2u_{i,t+\Delta t} + u_{i+1,t+\Delta t}\right]\right\} \qquad 2$$

where ϕ is a time weighting parameter.

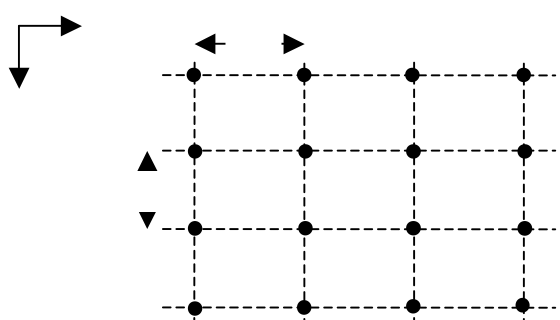

Figure 1. Discretization of consolidation equation.

2.1 Explicit solution

The explicit solution uses a value of $\phi = 0$ and calculates the pressure at time $t+\Delta t$ for each node sequentially. Initial excess pore pressures at time t are calculated for all nodes based on the applied stress. The pore pressure at time $t+\Delta t$ for node i is then calculated by setting $\phi = 0$ in equation 2 and rearranging to give:

$$u_{i,t+\Delta t} = u_{i,t} + \frac{c_v \Delta t}{(\Delta h)^2}\left(u_{i-1,t} - 2u_{i,t} + u_{i+1,t}\right) \qquad 3$$

This can be solved in a time-marching fashion since the pressure at time $t+\Delta t$ is only dependent upon the pressures at time t.

A well-known constraint of the explicit solution is that the timestep, Δt, must be small to ensure stability:

$$\Delta t < \beta \frac{(\Delta h^2)}{c_v} \qquad 4$$

(Abid and Pyrah, 1988) where β is a dimensionless factor that must be less than 0.5.

2.2 Implicit solution

To perform the implicit analysis, ϕ is set to 1 and equation 2 simplifies to.

$$u_{i,t+\Delta t} = \frac{u_{i,t} + \frac{c_v \Delta t}{(\Delta h)^2}\left(u_{i-1,t+\Delta t} + u_{i+1,t+\Delta t}\right)}{1 + 2c_v \Delta t/(\Delta h)^2} \qquad 5$$

Since the solution for pressure for a given position at time $t+\Delta t$ depends on the pressures at adjacent locations at $t+\Delta t$, the equation must be solved either by iterating or by simultaneously solving for all pressures at $t+\Delta t$ using matrix inversion. The system is unconditionally stable, regardless of Δt. However for small values of Δt the solution is oscillatory. Therefore to obtain good results:

$$\Delta t > \frac{1}{3}\frac{(\Delta h)^2}{c_v} \qquad 6$$

(Vermeer and Verruijt, 1981). Abid and Pyrah (1988) also suggest that there is an upper limit to the timestep and that if this is violated then results will become inaccurate. The maximum timestep suggested is on the order of 1.5 times the minimum (equation 6). Unlike the explicit solution, it is possible to gradually increase the timestep as large pressure gradients dissipate (Booker and Small, 1975).

2.3 Layered material

Most models will not have a constant value for Δh or c_v since different material layers will be present. To account for this, the equations are adjusted slightly. All u_{i+1} terms are multiplied by a factor α_i (Duncan et al., 2003):

$$\alpha_i = \frac{k_i}{k_{i-1}}\frac{h_{i-1}}{h_i} \qquad 7$$

where k_{i-1} , k_i are the permeabilities of the sublayers above and below the node and h_{i-1} , h_i are the thicknesses of the sublayers above and below the node.

All u_i terms are split into two pieces so for example, in the explicit solution (equation 3), the term for $u_{i,t}$ becomes:

$$\left(1-\frac{2c_v\Delta t}{(\Delta h)^2}\right)u_{i,t} \quad\rightarrow\quad \left(0.5+0.5\alpha_i-\frac{(c_v)_i\Delta t}{(\Delta h_i)^2}-\alpha_i\frac{(c_v)_{i+1}\Delta t}{(\Delta h_{i+1})^2}\right)u_{i,t}$$

This splitting of terms can be avoided by formulating the problem with a finite element scheme instead of finite difference (see Desai and Johnson, 1972).

The values of permeability depend on the material stiffness, which depends on the material type. For example, in a linear material:

$$k = c_v m_v \gamma_w \qquad 8$$

where m_v is the 1D compressibility, and γ_w is the unit weight of pore water. Similar expressions exist for non-linear material.

3 AN EXAMPLE MODEL

Lambe and Whitman (1979) present an example settlement problem as shown in Figure 2. It is assumed that the bottom sand layer is very stiff and permeable so we can simulate this with a 2-layer model that is drained at the top and bottom.

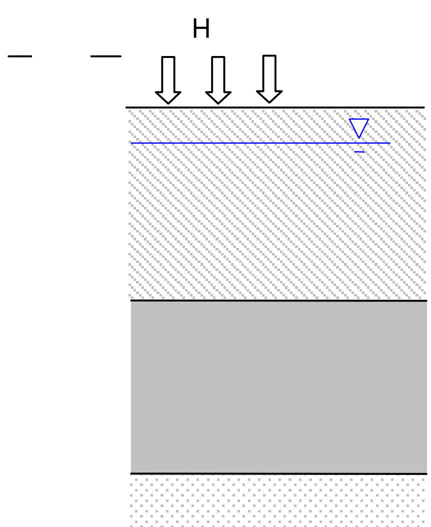

Figure 2. Two-layer model from Lambe and Whitman (1979, Example 27.2).

3.1 Explicit solution

To solve the problem using a finite difference approach, we need to discretize the problem. Coduto (1999) recommends at least 50 subdivisions for an accurate solution. This produces sub-layers with a thickness of

Δh ~ 0.19 m.

We can now calculate the maximum stable timestep according to equation 4. If we assume β = 0.25 (half the recommended maximum to ensure accuracy) and we use c_v = 946 m^2/year (silt), then the maximum stable explicit timestep according to equation 4 is:

$\Delta t_{explicit} < 9.54\times10^{-6}$ years

Because of the low permeability clay layer beneath the silt, the system takes several years to fully consolidate. Lambe and Whitman (1979) show that 99% consolidation occurs after 10.9 years. To obtain this solution therefore we need to execute over 1 million timesteps. Now multiply this times the number of models you wish to run simultaneously (if, for example, you have a load of finite extent and you wish to calculate differential settlement). The computation will require a significant amount of time even on modern computers.

It would be possible to increase the minimum timestep and therefore decrease the number of required steps by increasing the element size (equation 4). The timestep is very sensitive to the element size since the timestep depends on the element size squared. However, increasing the element size (and decreasing the number of divisions) will sacrifice accuracy in the solution.

The excess pore pressures calculated using the explicit method are shown in Figure 3 at different times.

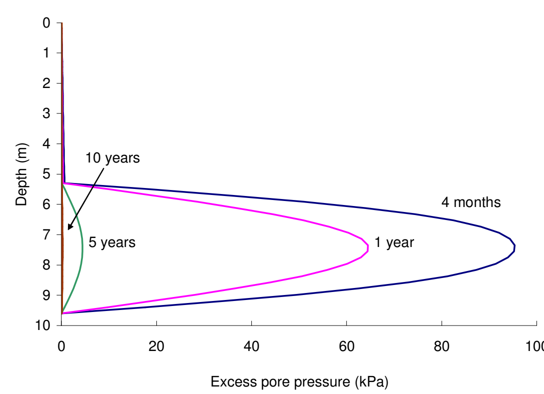

Figure 3. Excess pore pressures calculated using the explicit method.

3.2 Implicit solution

Equation 6 shows how the timestep in an implicit analysis must be greater than some minimum value to obtain reliable results. Vermeer and Verruijt (1981) suggest that the minimum timestep should be calculated by considering the sub-layer *next the draining boundary*. For the model in Figure 2, both the top boundary and bottom boundary are draining. We can therefore calculate two possible minimum timesteps:

$\Delta t_{implicit} > 1.27\times10^{-5}$ years (silt)
$\Delta t_{implicit} > 9.55\times10^{-3}$ years (clay)

If we use the smallest of these, then we get good results as shown in Figure 4. However, to run the solution to 99% consolidation (10.9 years) requires 858,000 timesteps and is therefore not much faster than the explicit solution

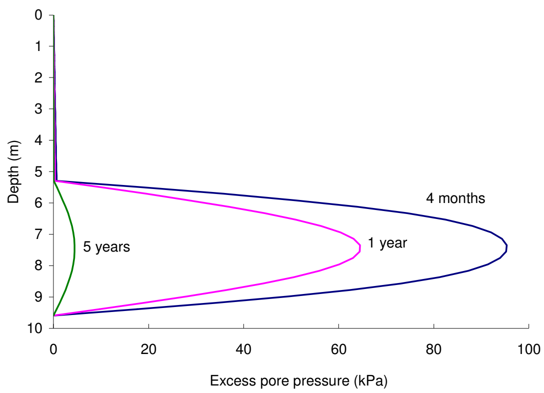

Figure 4. Excess pore pressures calculated using the implicit solution and a timestep of 1.27×10^{-5} years.

If we use the larger timestep, we get the results shown in Figure 5. It can be seen that the pressures at early times are inaccurate. Recall that the initial timestep should be less than about 1.5 times the calculated minimum to obtain accurate results. Therefore it appears that the timestep should be between 1.27×10^{-5} and 1.9×10^{-5}.

Booker and Small (1975) describe how stability and accuracy can be maintained by starting with a small timestep and increasing the timestep gradually as the solution progresses. This will substantially speed up the solution and therefore this is the approach adopted for all future calculations in this paper. It may be tempting to try the same approach with the explicit solution, however there is no guarantee that the solution will remain stable and accurate. Instability develops once equation 4 is violated, even if gradients are small.

The only problem with the implicit approach is that you may want to see results at early times that cannot be calculated because of the minimum timestep stipulation (equation 6). The next section outlines a hybrid scheme that will overcome this problem.

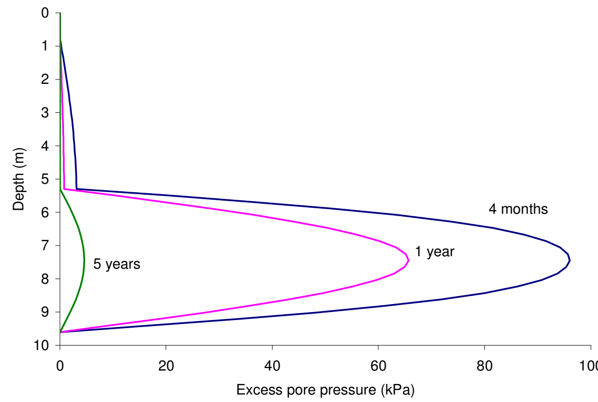

Figure 5. Excess pore pressures calculated using the implicit solution and a timestep of 9.55×10^{-3} years.

4 HYBRID SOLUTION

Section 3 showed that using the explicit or implicit solution alone may be either slow or inaccurate. These problems can be overcome by combining the two methods in the following way:

1. Compute the minimum implicit timestep, $\Delta t_{implicit}^{\min}$ according to equation 6
2. Numerical experiments have shown that implicit results are inaccurate if less than 5-10 timesteps are executed. Therefore we will run an explicit computation for a length of time equal to $10\times \Delta t_{implicit}^{\min}$
3. Use the explicit results as the initial conditions for an implicit computation using the timestep calculated in step 1.
4. Gradually increase the timestep as the calculation continues (see Booker and Small, 1975). Since a matrix has to be constructed and inverted every time the timestep changes (equation 5), it would be very slow to increase the timestep every step. Through a series of numerical experiments, it was found that a good balance could be obtained by increasing the timestep by a factor of 1% every 20 steps.

4.1 Pore pressure example

For the model shown in Figure 2, $\Delta t_{implicit}^{\min} = 1.27\times10^{-5}$ years or 6.7 minutes. We know that we won't get accurate results from the implicit analysis until ~ 10 steps have been executed. Therefore if we were to perform a purely implicit analysis, then we cannot get good results at times less than 67 minutes. If you wanted to see the pore pressures one hour after loading, it would be impossible. (This is unlikely, but it wouldn't be hard to come up with a scenario where the minimum implicit time was actually quite long). Therefore we will use the hybrid approach, starting the computation with an explicit analysis. The computation would unfold as follows:

- A minimum implicit timestep of $\Delta t_{implicit}^{\min}$ = 6.7 minutes is calculated.
- An explicit calculation is performed until the model reaches a time of $10\times\Delta t_{implicit}^{\min}$. For an explicit timestep of 5 minutes, this requires 14 steps.
- The implicit computation is then performed starting from 14×5 = 10 minutes until consolidation has finished 10.9 years). The implicit timestep is increased by a factor of 1.01 every 20 steps. This requires < 550 time steps.

Clearly this approach is much more efficient that performing an exclusively explicit analysis (~600 time steps rather than 1 million) and yet early time results can be accurately obtained, which is not the case for a purely implicit analysis (see Figure 6).

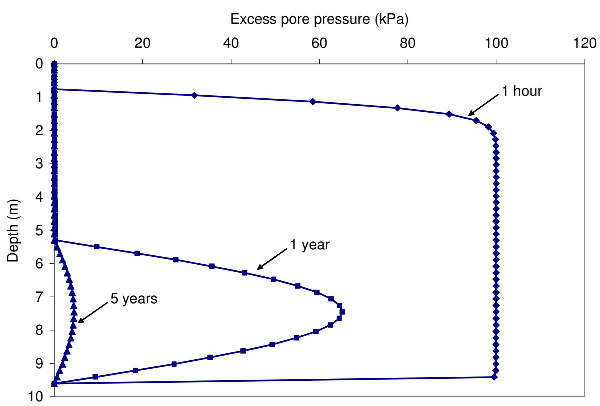

Figure 6. Comparison of excess pore pressures calculated using the hybrid solution scheme.

4.2 Consolidation examples

Zhu and Yin (2005) provide analytical solutions for consolidation in two-layer soil columns. The geometry of the problems is shown in Figure 7. Where:

- c_{v1} , c_{v2} are the coefficients of consolidation
- m_{v1} , m_{v2} are the 1-D compressibilities
- H_1 and H_2 are the thicknesses of the two layers

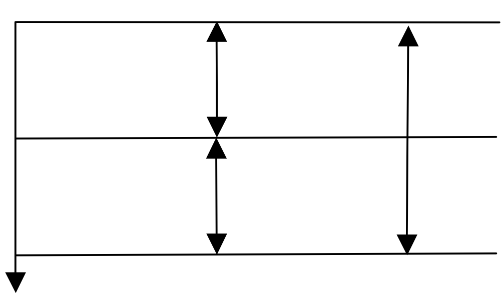

Figure 7. Geometry of two layer consolidation problems.

Three problems were simulated using the parameters of Table 1. In all models m_{v1} = m_{v2} = 0.001 m^2/kN. All models were discretized into 100 sub-layers. A load of 10 kPa was applied to the top, and the bottom was held fixed.

Table 1. Parameters defining the 2-layer models.

Model	H_1 *(m)*	H_2 (m)	c_{v1} (m^2/year)	c_{v2} (m^2/year)	*Drainage*
1	4.737	10	1	361	Double
2	10	2.967	102.23	1	Single
3	0.330	10	1	102.23	Single

Analytical solutions in Zhu and Yin (2005) are for the degree of consolidation, rather than the excess pore pressures. Degree of consolidation was calculated by first determining the *final* settlement in each model. This was computed by simply multiplying $m_v \times$ *loading stress* $\times$ *H* for each layer and adding them. The models were then run and degree of consolidation was calculated at different times by dividing the consolidation settlement at that time by the final consolidation settlement. To compute settlement at any given time the following steps are followed:

- The stress in each sub-layer is determined by subtracting the excess pore pressure from the applied load.
- The strain in each sub-layer is the stress multiplied by m_v.
- The displacement in each sub-layer is the strain multiplied by the sub-layer thickness
- The settlement is the sum of all sub-layer displacements.

Results for the hybrid computation scheme are shown in Figure 8.

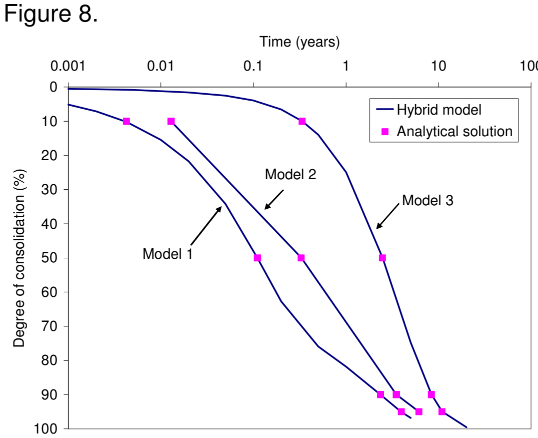

Figure 8. Results for the three two-layer models computed using the hybrid calculation scheme

Zhu and Yin (2005) give the analytical solution for consolidation times to 10%, 50%, 90% and 95% consolidation as shown in Figure 8. If these values are compared with the values calculated by the numerical models, then errors in the models can be calculated.

Table 2 shows the errors in the hybrid solution as well as errors for a purely explicit solution. The hybrid solution and the explicit solution give virtually the same results – the only difference being that the hybrid solution performs the calculations much faster.

Table 2. Errors (%) in the numerical models compared with the analytical solution of Zhu and Yin (2005).

	Model 1		Model 2		Model 3	
	Avg.	Max.	Avg.	Max.	Avg.	Max.
Hybrid	0.53	1.7	0.048	0.12	0.091	0.27
Explicit	0.56	1.8	0.053	0.10	0.086	0.27

5 FIELD EXAMPLE

Huang et al (2006) present a study of the consolidation of an embankment on soft clay in Eastern Australia called the "Teven Road trial embankment". The problem is interesting because there is a layer of dense, high permeability sand in between two clay layers.

5.1 The model

A model of the Teven Road trial embankment was constructed using Settle3D (Rocscience, 2007) to test the hybrid consolidation calculation scheme.

The geometry of the model is shown in Figure 9. The material parameters used are shown in Table 3 where e_0 is the initial void ratio, OCR is the overconsolidation ratio, C_c is the compression index, C_r is the recompression index and k is the permeability.

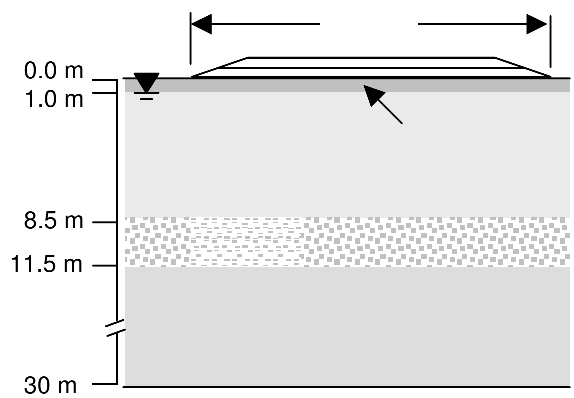

Figure 9. Model geometry of Teven Road trial embankment.

Table 3. Soil parameters used in the model.

Layer	e_0	OCR	C_c	C_r	k (m/year)
Crust clay	0.8	8	0.18	0.025	2.2
Soft clay	2.3	2.4	1.35	0.046	0.051
Sand	0.65	1	0.009	0.009	31.4
Stiff clay	1.7	1.03	0.31	0.021	0.022

The embankment was built up to a height of 1.5 m in three stages. A thickness of 0.5 m was added at 0 days, 40 days and 65 days.

Five piezometers were installed at the actual site to measure pore pressures during consolidation. To compare the model results to the field measurements, one-dimensional consolidation calculations were performed at the same 5 locations in the model (see Figure 10). Each one-dimensional calculation 'string' consisted of 60 divisions. The divisions were not evenly spaced along the string – the discretization was denser at material boundaries to account for higher pore pressure gradients at these locations.

Three dimensional stresses due to the embankment loads were computed by integrating the Boussinesq solution (see Rocscience, 2007). These stresses were then used to set the initial excess pore pressures at each point in each 1D string. The hybrid calculation scheme was then used to compute excess pore pressures at several times after the application of the load(s). The coefficient of consolidation was computed for each sublayer in the string with the following equation:

$$c_v = \frac{2.3(1+e_0)\sigma_i' k}{C_c \gamma_w} \qquad 9$$

Where σ_i' is the initial effective stress. For overconsolidated material, C_c is replaced by C_r.

Figure 10. Top view of embankment showing location of piezometers/consolidation calculations.

5.2 Model results

Excess pore pressures for string #1 (the centre of the embankment) are shown during the construction stage in Figure 11 and during consolidation in Figure 12. For each stage of construction, the excess pore pressure essentially mirrors the loading stress due to the embankment. After construction has finished, the pore pressures gradually dissipate to zero at about 140 years.

In Figure 12 the presence of the high-permeability sand layer is clearly indicated by the 'step' in the curve showing rapid dissipation of pore pressure in this layer. The sand layer essentially acts as a short-circuit for fluid flow connecting the clay layers above and below.

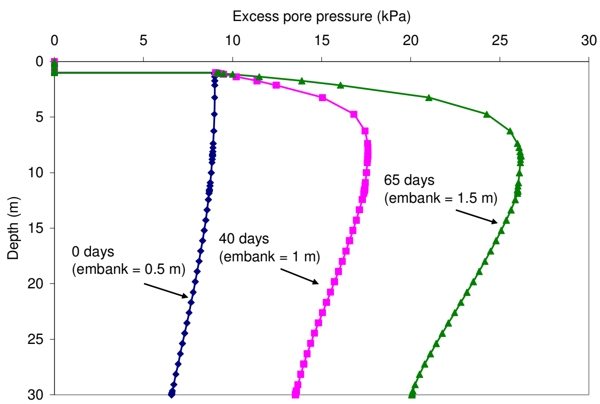

Figure 11. Excess pore pressure under the centre of the embankment over the first 65 days.

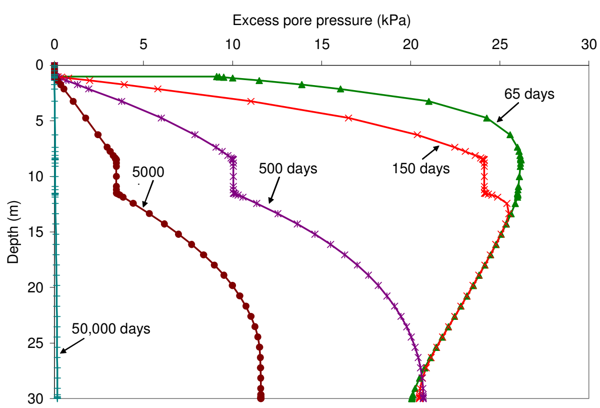

Figure 12. Excess pore pressure under the centre of the embankment after construction (from 65 days to ~ 137 years).

Figure 13 compares results to a two-dimensional finite element model and the actual field data at a point 4 metres below the centre of the embankment. The results from the one-dimensional hybrid computation scheme match reasonably well with the finite element and field data. The early jump in pore pressure observed in the 1D solution is probably seen because each stage of embankment construction was performed instantly in this model, whereas in the finite element model (and in reality) the embankment construction occurred gradually over several days. Both the finite element model and the 1D model show slower pore pressure dissipation than was observed in the field. This suggests that the permeabilities used in the models were probably too low. The discrepancy may also be due to the horizontal flow not accounted for in the models.

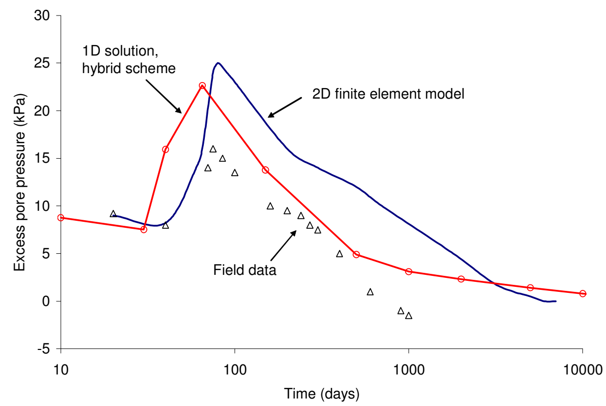

Figure 13. Excess pore pressure at a point 4 m below the centre of the embankment. Finite element and field data results from Huang et al (2006).

5.3 Comparison with explicit solution

The purely explicit solution gives virtually identical results to those from the hybrid calculation scheme. The main difference is the time required to reach the solution. To compute the results for the 5 strings took approximately **four hours** on a single processor PC (3.2 GHz Pentium 4). The hybrid solution took less than **30 seconds**. The large time required for the explicit solution is due to the high permeability sand layer, but also due to the very small element sizes near the material boundaries. The solutions could be sped up by increasing the element size (see equation 4) but some accuracy may be lost.

6 SUMMARY

The one-dimensional consolidation equation is often used to compute excess pore pressures in geotechnical engineering consolidation problems. For multi-layered systems, an explicit finite difference approach is often used to compute the solution. We have shown that for soil profiles with large permeability contrasts, the explicit solution can be prohibitively expensive in terms of computer resources required.

The equation can be solved much faster using an implicit approach, however a minimum timestep is required to obtain accurate solutions. If results are desired at early times, then the implicit solution cannot provide them.

By combining the explicit and implicit solution approaches into a hybrid scheme, accurate results can be obtained quickly. This is demonstrated by using the new approach to simulate consolidation in two-layer and three-layer soil profiles. It is shown that the hybrid approach combines the speed and accuracy of the implicit approach with the versatility of the explicit method.

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