Introduction of inherent anisotropy into a critical-state constitutive model for sands

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ABSTRACT
A critical state constitutive model for sand was previously developed with emphasis on capturing the main aspects of the behavior of loose liquefiable sands. The model, which was presented in details in previous publications, was formulated and verified for various drained and undrained monotonic and cyclic loadings of sands. However, since in-situ soils often exhibit strong inherent anisotropy, the model was extended to predicting the behavior of such soils. In this paper, it is shown that adding a new parameter similar to that proposed by Li and Dafalias enables the model to simulate the behavior of inherently anisotropic sands.

RÉSUMÉ
Un état critique modèle constitutif pour le sable a été développé à l'accent étant mis sur la capture des principaux aspects du comportement des sables lâches liquéfiables. Le modèle, qui a été présenté en détails dans les publications antérieures, a été formulé et vérifiés pour diverses drainés et non drainés chargements monotone et cyclique des sables. Toutefois, depuis les sols in situ présentent souvent une forte anisotropie inhérente, le modèle a été étendu afin de lui permettre de prédire l'anisotropie telles. Dans cet article, il est démontré que l'ajout d'un nouveau paramètre similaire à celle proposée par Li et Dafalias permet au modèle pour simuler le comportement des sables nature anisotrope.

Keywords: Sand anisotropy, sand modeling, critical state model, constitutive model, inherent anisotropy

1 INTRODUCTION

Many numerical analyses carried out in the field of geotechnical engineering require the use of a constitutive model. Constitutive models enable researchers to perform analysis and design for a large range of geotechnical problems without the need for the time and cost associated with conducting many laboratory tests.

Anisotropic behavior of soils, especially sands, is an important and often challenging aspect of soil constitutive modeling. This aspect of soil behavior has been investigated in many experimental studies carried out by various researchers (see e.g. Miura and Toki 1984; Yoshimine 1996). As these experimental studies have shown, and also pointed out by Been et al. (1991), anisotropy has significant effects on the behavior of sands.

Yoshimine (1996) preformed a series of hollow cylinder (HC), undrained tests on Toyoura sand. In these tests, the direction of applied principal stresses with respect to the direction of soil deposition, and also the ratios between the principal stresses were kept constant in each test. Test results showed that the soil stress-strain behavior and stress path up to the critical-state failure were significantly affected by the direction of the applied principal stresses relative to the orientation of soil deposition, and therefore, they demonstrate the significance of anisotropy on sand response to loading.

A critical-state constitutive model for sands was presented by Imam et al. (2005). This model was used to simulate sand behavior as observed in the triaxial compression and extension loading. However, in order to predict the behavior in other modes of shearing such as those conducted in the hollow cylinder apparatus, some extensions need to be done to the model formulation such that effects of anisotropy and mean effective principal stress can be properly accounted for. In this paper, a relatively simple approach is used for taking into account the effect of material-inherent anisotropy on the yielding of sands, and this is incorporated into this model. In this approach, which was used by Li and Dafalias (2002) to incorporate anisotropy into their constitutive model, an anisotropy state parameter, *A*, is added to the model parameters. A second-order fabric tensor F_{ij} (Oda 1999), defined based on micromechanical behavior of sand particles, is employed to describe the material-inherent anisotropy, and the scalar-valued state variable *A* is used to define this tensor using joint invariants of the stress tensor σ_{ij} and the fabric tensor F_{ij}.

3 THE ORIGINAL CONSTITUTIVE MODEL

A critical state constitutive model for sands was developed with emphasis on taking into account important aspects of the behaviour of loose liquefiable sand. Details of the model and its formulation are described elsewhere (see Imam et al. 2005), and the constitutive relationships are summarized in the Appendix of this paper. The model uses a capped yield surface with the stress ratio M_p at its point of peak deviatoric stress (q) depending on the sand material properties.

In the model, stress-dilatancy is based on Rowe's (1962) relationship combined with a modified form of the Manzari and Dafalias (1997) equation. The failure criterion is expressed in terms of a friction angle that depends on the current state parameter ψ through a slightly modified version of the Wood et al. (1994) relationship.

The model uses a single set of parameters to predict sand behavior over a wide range of void ratios and confining pressures. The critical state line represents soil state at large strain, while the behavior at small and medium strains is captured by other material parameters such as the yielding, dilatancy, and plastic modulus parameters, which take into account anisotropy.

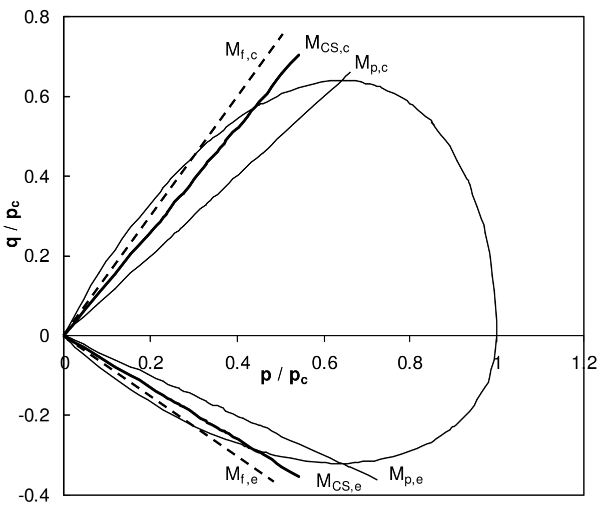

Figure 1 – Yield surface of isotropically consolidated sand

Figure 1 shows a graphical representation of the yield surface, and the stress ratios at critical state (M_{cs}) and at failure (M_f) in a p-q plane normalized to the maximum mean normal stress at yielding (p_c). Values of M_p in triaxial compression and triaxial extension are referred to as $M_{p,c}$ and $M_{p,e}$, respectively. These stress ratios control the yield surface shape (i.e. width), and account for effects of void ratio, mean normal stress, and inherent anisotropy on the yielding stresses. A small M_p results in a slender yield surface and applies to sand that is loose, or subjected to high confining pressures, or loaded in a weak direction such as the triaxial extension. Stress-induced anisotropy is represented by stress ratio α, at which the tangent to the yield surface is parallel to the q-axis. This stress ratio is non-zero only in anisotropically consolidated sand. Size of the yield surface is determined by p_c. In anisotropically consolidated sand, the stress state corresponding to this maximum mean normal stress does not lie on the p axis since it is associated with a shear stress.

2 VARIATIONS OF STRESS RATIO M_P

Based on suggestions by previous researchers and comparisons of test results, Imam et al. (2002), indicated that in loose and very loose sands, the ratio of shear stress to mean normal stress at the peak point of the undrained effective stress path (UESP) is very close to the stress ratio at the peak point of the capped yield surface, M_p, and therefore, the latter can be estimated from measurements of the UESP, and used in the formulation of the yield surface. Since formulation of the original model is such that the stress ratio M_p affects a number of aspects of the predicted response such as the yielding stresses, plastic modulus, etc (see Imam et al. 2005), modifying the original model, the influence of anisotropy on the stress ratio M_p is investigated, and this effect is incorporated into the model. Ability of the modified model to account for sand anisotropy is then examined.

2.1 Effects of anisotropy on M_p

In the extensive experiments conducted by Yoshimine (1996) using the HC apparatus, effects of the intermediate principal stress (accounted for through b= $(\sigma_2- \sigma_3)/(\sigma_1- \sigma_3)$) and the direction of loading (accounted for through the angle α between the direction of major principal stress and normal to the bedding planes, as illustrated in Fig. 1(a)) were investigated. In these series of undrained tests, all specimens were consolidated isotropically to 100 kPa before being sheared.

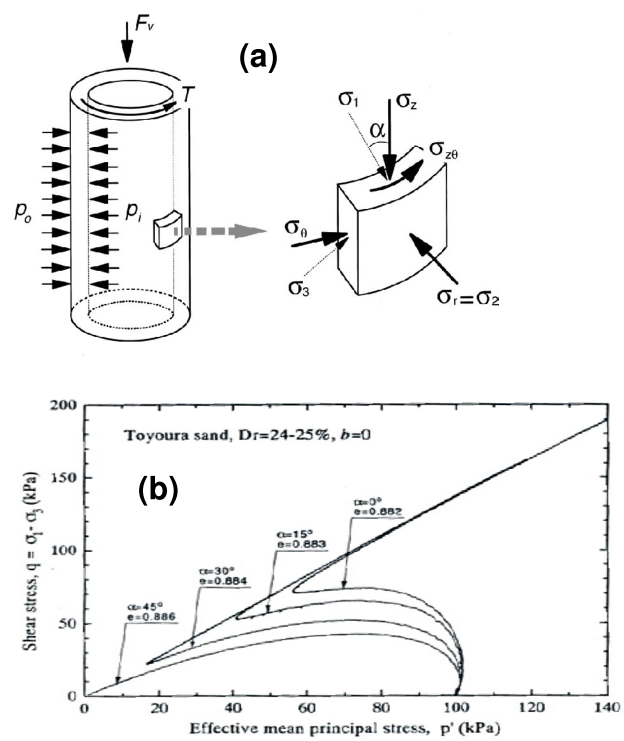

Figure 1.Loading in HC apparatus (after Yoshimine 1996)

Figure 1 (b) shows stress paths for a sample of such test series, in which all tests were conducted at b=0, but the angle α was varied from zero to 45 degrees. It may be seen from this figure that although the relative density and b are the same for all the tests, the stress ratio q/p at which the peak value of q is reached is not constant but it changes with α.

It is noted that considering all possible conditions of loading, b can vary from 0 to 1 and α from 0 to 90 degrees. Loading in triaxial compression (TC) corresponds to b=0 and α=0 and in Traxial extension (TE) to b=1 and α=90. In his HC testing, Yoshimine (1995) selected certain combinations of b and α such that unacceptable non-uniformities will not occur in the soil sample.

Fig. 2 shows a schematic representation of typical values of stress ratios $M_{p,c}$ and $M_{p,e}$ obtained for the two extreme conditions of TC and TE loading respectively.

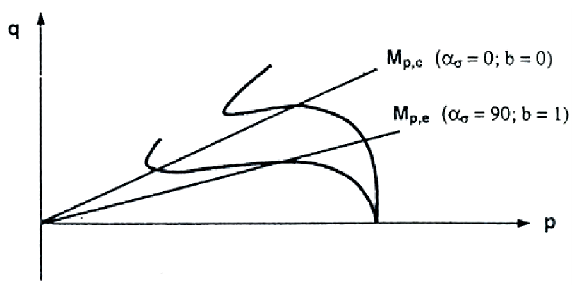

Figure 2.Shematic representation of the peak point of UESP in TC and TE tests (Imam et al. 2002)

In expressing the stress state at the peak point of the UESP for use in sand modeling, it is preferable to use $\sin\varphi_p$ in which φ_p is the mobilized friction angle, rather than the stress ratio M_p (Imam et al 2005). These two stress variables are related to each other through the following equation:

$$M_p = \frac{6\left(1-b+b^2\right)^{1/2}\sin\varphi_p}{3+(2b-1)\sin\varphi_p} \qquad [1]$$

Fig. 3 shows variations of $\sin\varphi_p$ with void ratio obtained from experiments with various combinations of b and α. These results indicate that as α increase, $\sin\varphi_p$ decreases and, in most cases, as b increases, $\sin\varphi_p$ slightly decreases. They are consistent with other studies reported by different researchers (e.g. Shibuya and Hight 1987). Moreover, Fig. 3 shows that for a certain void ratio, values of $\sin\varphi_p$ obtained from the combinations of b and α corresponding to TC and TE constitute the upper and lower extremes respectively, and that the lines connecting values of $\sin\varphi_p$ for the same combination of b and α have approximately the same slope of variation with void ratio. In the following sections, these variations are formulated using the fabric tensor and parameter A described briefly before.

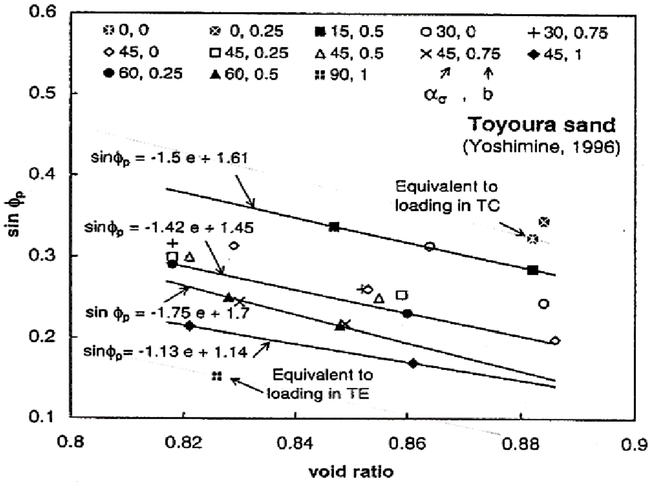

Figure 3. Variations of $\sin\varphi_p$ with void ratio for various combinations of b and α (modified after Imam et al. 2002)

3 FORMULATION OF THE VARIATION OF M_P

3.1 The Fabric Tensor F_{ij}

Random distribution of non-spherical particles such as those of sand has certain statistical characteristics for the package of particles in their spatial arrangement. These characteristics are known as material fabric (Brewer 1964; Oda 1972). The orientation of non-spherical sand particles can be determined by means of a pair of unit vectors, n and –n along their major axis of elongation. Oda (1999) defined a second order tensor F_{ij}, named the fabric tensor as follows:

$$F_{ij} = \frac{1}{2N}\sum_{k=1}^{N} n_i^k n_j^k \qquad [2]$$

in which N is the number of particles in a specific volume and n_i^k and n_j^k are components of the kth vector. Components of this tensor show the net portion of particles that are oriented in a specific direction. F_{IJ} is symmetric and it therefore can be represented by its principal values. In most cases, soils are transversely isotropic and, as a result, two of the principal values are equal to each other. According to Eq. 2, F_{IJ} has a unit trace and, for the principal values F_1, F_2, and F_3, since $F_2=F_3$ we have $F_1=1-(F_2+F_3)=1-2F_3$. This means that for a transversely isotropic soil with known direction of deposition – which is usually the vertical direction– just one scalar quantity is needed to define the fabric tensor.

For such a material, F_{IJ} can be written as (Oda and Nakayama 1988):

$$F_{ij}^{'} = \frac{1}{3+\Delta}\begin{pmatrix} 1-\Delta & 0 & 0 \\ 0 & 1+\Delta & 0 \\ 0 & 0 & 1+\Delta \end{pmatrix} \quad [3]$$

In which Δ is a measurable quantity which represents the magnitude of anisotropy of the particles. It can be shown that Δ is equal to zero when the material is isotropic and changes to unity when all particles are oriented in one plane (plane of transverse isotropy). This parameter depends on sand particle shapes and the process of soil deposition.

3.2 Anisotropy state parameter A

Li and Dafalias (2002) used a relatively simple approach to account for the effect of the relative orientation of the stress and fabric tensors. A scalar-valued state variable A was introduced, which is a function of the joint invariants of F_{ij} and σ_{ij}. A normalized modified stress tensor $\tilde{T}_{ij}$ was defined using the aforementioned fabric tensor as follows:

$$\tilde{T}_{ij} = \left(\hat{\sigma}_{ik}F_{kj}^{-1} + F_{ik}^{-1}\hat{\sigma}_{kj}\right) \quad [4]$$

In which the normalized stress tensor $\hat{\sigma}_{ij}$ is defined such that its mean normal stress equals unity. The tensor $\tilde{T}_{ij}$ is then used for the definition of all stress-related variables. The anisotropic state variable is defined as:

$$A = \frac{\tilde{R}}{M_c g(\tilde{\theta})} - 1 \quad [5]$$

In which $\tilde{R} = \sqrt{3\tilde{r}_{ij}\tilde{r}_{ij}/2}$ with $\tilde{r}_{ij} = \tilde{s}_{ij}/p$ is a normalized modified stress ratio invariant, $g(\tilde{\theta})$ is an interpolation function that interpolates $\tilde{R}$ according to the normalized modified Lode angle $\tilde{\theta} = -\frac{1}{3}\sin^{-1}\left(9\tilde{r}_{ij}\tilde{r}_{ij}\tilde{r}_{ij}/2\tilde{R}^3\right)$ and:

$$g(\tilde{\theta}) = \frac{2c}{(1+c)-(1-c)\cos 3\theta} \quad [6]$$

$$c = \frac{M_e}{M_c} \quad [7]$$

and M_c and M_e are the critical state stress ratios in TC and TE respectively.

It can be shown that for isotropic materials A=0, and for anisotropic materials, depending on the orientation of the stress relative to that of the soil fabric, it can be negative or positive. The variation of parameter A for selected values of Δ=0.2, M_c=1.25 and c=0.75 with b and α is shown in Fig. 4. As shown in Fig.4 parameter A decreases with both b and α and it is more sensitive to α than to b. Moreover, the TC and TE modes of shearing are consistent with the highest and lowest values of A respectively. Considering the observed effects of b and α on $\sin\varphi_p$ discussed before, it can be seen that the variation of A is consistent with the variation of $\sin\varphi_p$ with b and α.

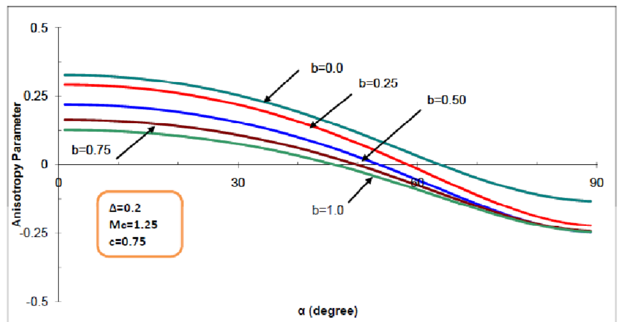

Figure 4. Variation of A with α and b

4 MODIFICATION OF THE CONSTITUTIVE MODEL

Modification of the formulation of the original model for the calculation of M_p is described here in detail. Details of the original model formulation can be found in the Appendix. As indicated before, some aspects of the model such as the yielding stresses and plastic modulus are affected by this stress ratio through the model formulation. These will become a function of anisotropy once M_p is made a function of anisotropy. However, other aspects such as the stress ratios at failure and critical state are not affected by M_p and are, therefore considered to remain unaffected by anisotropy. The independence of these stress ratios of sand anisotropy has also been observed in experiments reported by a number of researches (see e.g. Yamada and Ishihara 1979; Cambu and Lanier, 1988) and also assumed in some recent constitutive models for sands (e.g. Manzari and Dafalias 1997; Li and Dafalias 2002)

The stress ratio M_p for any soil state can be obtained using Eq. [1] for which the friction angle φ_p is determined from the following equations if loading is in TC and TE:

$$\sin\varphi_{p,c} = \sin\varphi_\mu - k_p\psi_p \quad [8]$$

$$\sin\varphi_{p,e} = \sin\varphi_\mu - k_p\psi_p - a_p \quad [9]$$

In which φ_μ is the friction angle at ψ_p=0 and, in TC, it is approximately equal to the inter-particle friction angle for the soil grains. The k_p and a_p are material parameters and

ψ_p is the state parameter at the peak point of the yield surface. These relationships can be used to calculate the stress ratio M_p for TC and TE. However, in order to determine this stress ratio for loading between TC and TE, and to account for the effects of inherent anisotropy, equations [8] and [9] are replaced by the following proposed equation:

$$\sin\varphi_p = \sin\varphi_\mu - k_p\psi_p - A_p(A) \qquad [10]$$

In which:

$$A_p = \frac{A_c - A}{A_c - A_e} \times a_p \qquad [11]$$

and A_c and A_e are the anisotropy parameters in TC and TE respectively, A is the anisotropy parameter for the current state and mode of shearing of the soil, and a_p is the material parameter used in the original model.

In Fig. 5, results calculated using the modified formulation are compared with measured values. It indicates that the variation of $\sin\varphi_p$ with b and α can be modeled with good accuracy using equation [10].

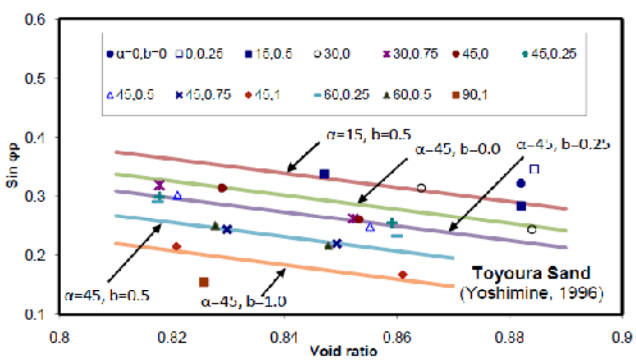

Figure 5. Observed and predicted variations of $\sin\varphi_p$ with b and α using Eq. [10]

Using Equation [10] to calculate $\sin\varphi_p$ for various combinations of b and α and, substituting this value into equation [1], stress ratio M_p can be calculated for various combinations of b and α and used for predicting sand behavior subjected to shearing at various combinations of b and α.

5 PERFORMANCE OF THE MODIFIED MODEL

Accuracy of the modified model in predicting the anisotropic behavior of sands is investigated here by comparing model predictions and laboratory results of the behavior of Toyoura sand subjected to various loading directions as presented by Yoshimine (1996).

Model parameters used for predicting the behavior of Toyoura sand are listed in table 1. The critical state line used for Toyoura sand is defined using the equation: $-0.0063477p^3+0.0367p^2-0.11991p+0.92548$ (p in MPa) as suggested by Imam et al. (2005).

Table 1. Model parameters for Toyoura sand

Parameter type	Parameter name	Value
Peak state	k_p	1.2
	Φ_μ	20
	a_p	0.45
Stress-dilatancy	Φ_{cs}	30
	k_{pt}	0.75
	a_{pt}	0.01
Plastic stiffness	h	1
Elasticity	G_a	5000
	k_a	8500
Anisotropy	Δ	0.2
	c	0.75
	M_c	1.25

Figures 6 and 7 compare predicted and observed responses of Toyoura sand consolidated to two different void ratios and subjected to loading with $b=0.0$, but with α varying from 0 to 45 degrees. Observed and predicted responses show a generally good match.

Measured and predicted responses of Toyoura sand loaded under b=0.25 and α varying from 0 to 30 degrees are compared in Figure 8. The two responses are generally similar. However, observed response at α = 30 degrees is somewhat softer than the predicted response. A reason for such difference might be the possible development of localized zones of higher strains in the actual test sample which might have led to increased softening compared to what would be observed in a uniformly straining sample. The predicted response, which is based on an average behavior for a single soil element, is not expected to match the behavior of a sample with non-uniform straining pattern.

6 SUMMARY AND CONCLUSIONS

A previously developed critical state model was extended to modeling the behaviour of inherently anisotropic sands, using an anisotropy parameter as defined by Li and Dafalias (2002). Comparisons of predicted and observed responses of Toyoura sand tested in HC indicated that model predictions is generally in good agreement with observed behaviour. However, for certain combinations of b and α, they have some difference, a possible reason for that is the non-uniformities and localized deformations that might develop in the real sample compared to the uniform straining that is assumed in modeling a single element soil.

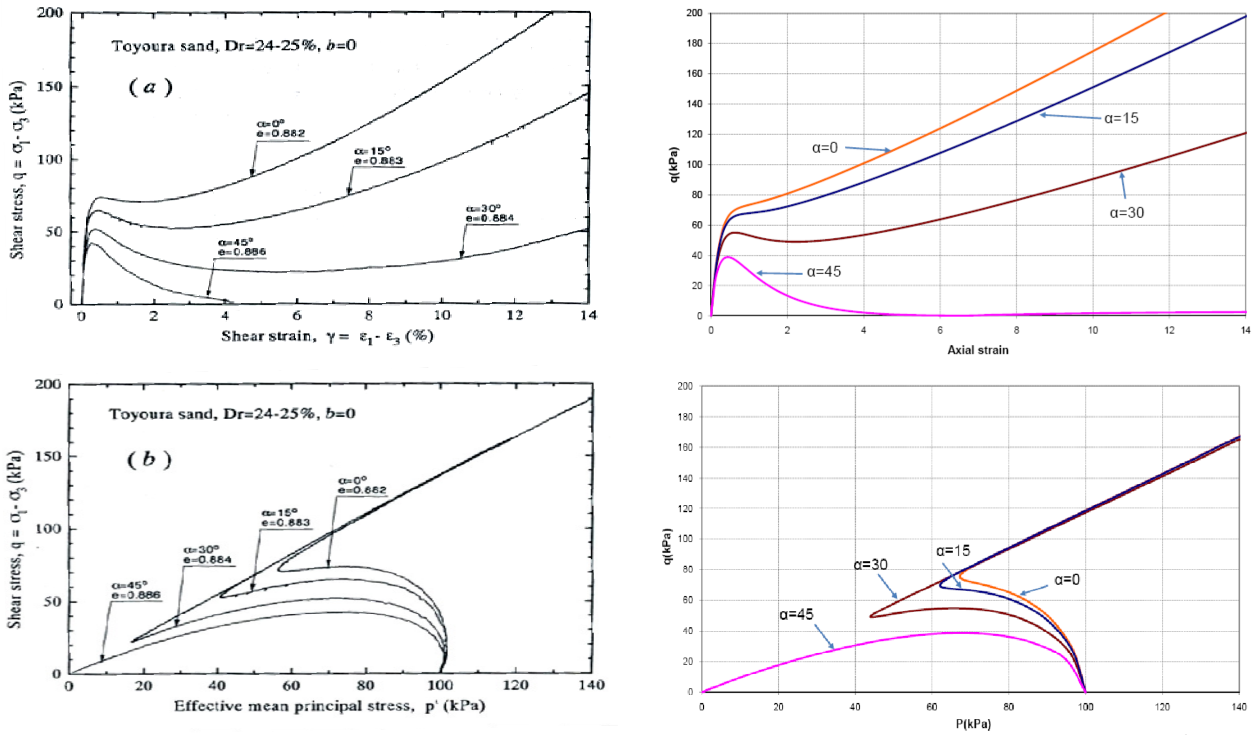

Figure 6. Predicted (right) and observed behavior of Toyoura Sand consolidated to D_r=24-25%, b=0.0 and α=0° to 45°.

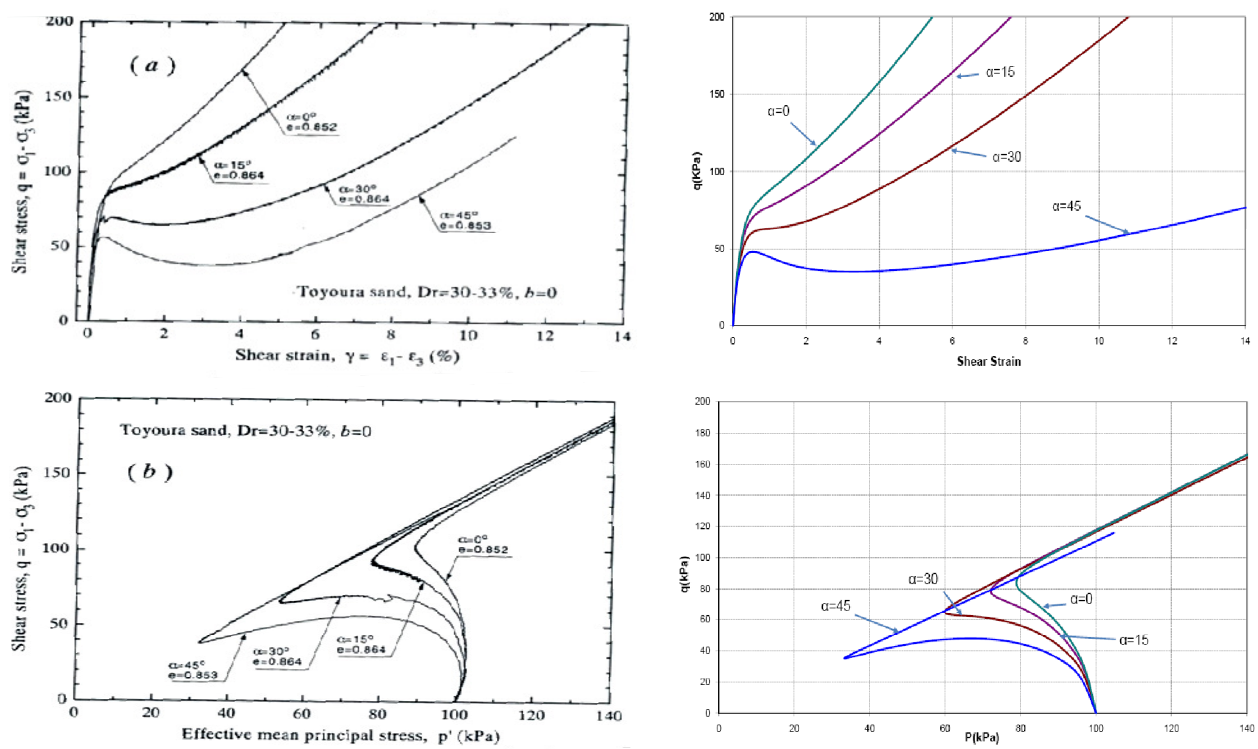

Figure 7. Predicted (right) and observed behavior of Toyoura Sand consolidated to D_r=30-33%, b=0.0 and α=0° to 45°.

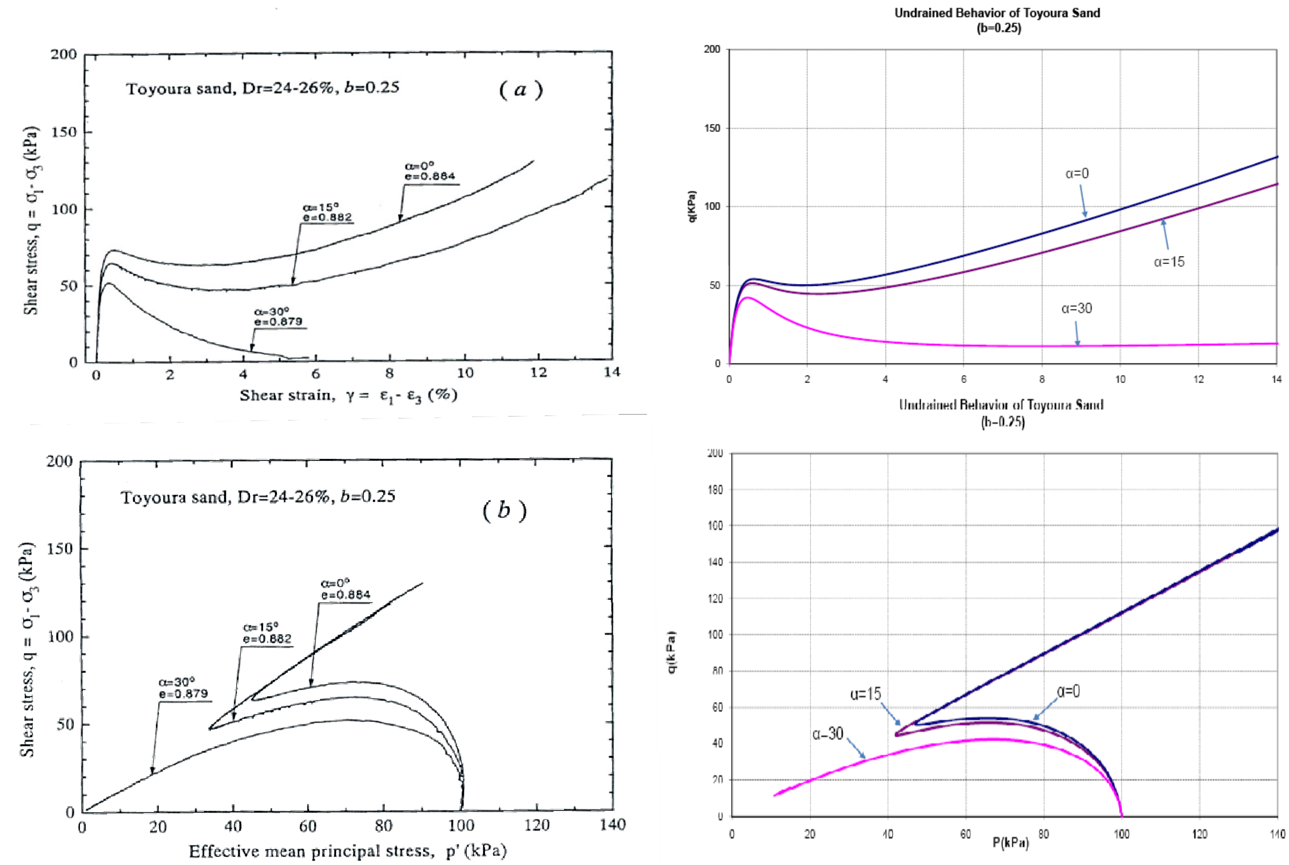

Figure 8. Predicted (right) and observed behavior of Toyoura Sand consolidated to D_r*=24-26%*, *b=0.25* and $\alpha=0^o$ *to* 30^o.

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APPENDIX

The yield surface is defined as:

$$f = (\eta - \alpha)^2 - M_\alpha^{\,2}\left[1-\left(p/p_c\right)^{\frac{1}{2}}\right] = 0 \qquad [1\text{-a}]$$

$$M_\alpha^{\,2} = (5M_p - \alpha)(M_p - \alpha) \qquad [2]$$

in which, for triaxial compression (TC) and triaxial extension (TE) we have:

$$M_{p,c} = \frac{6\sin\varphi_{p,c}}{3-\sin\varphi_{p,c}} \quad \text{in TC} \qquad [3a]$$

$$M_{p,e} = \frac{6\sin\varphi_{p,e}}{3+\sin\varphi_{p,e}} \quad \text{in TE} \qquad [3b]$$

and $\varphi_{p,c}$ and $\varphi_{p,e}$ are the friction angles at the point of peak q in TC and TE, respectively, and are obtained from:

$$\sin\varphi_{p,c} = \sin\varphi_\mu - k_p\,\psi \quad \text{in TC} \qquad [4\text{-a}]$$

$$\sin\varphi_{p,e} = \sin\varphi_\mu - k_p\,\psi - a_p \quad \text{in TE} \qquad [4\text{-b}]$$

in which φ_μ is the friction angle corresponding to $\psi_p = 0$ in TC and is typically close to the inter-particle friction angle of the sand; k_p and a_p are material parameters, and ψ is the state parameter. A Mohr-Coulomb type failure criterion, expressed in the following form, is used:

$$\sin\varphi_f = \sin\varphi_{cs} - k_f\,\psi \qquad [5]$$

in which φ_{cs} is the critical state friction angle and k_f is a material parameter which is taken to be 0.75 for sand loaded in both TC and TE. Friction angles obtained from [6] are converted to equivalent stress ratios at failure $M_{f,c}$ and $M_{f,e}$ for TC and TE as in [3]. These are the maximum stress ratios attainable at the current soil state, and may not be equal to the current stress ratio η. It is only at critical state (ψ = 0) where the current and failure stress ratios coincide and we have $\eta = M_f = M_{cs}$. The flow rule is determined from the following relationship:

$$d = \frac{d\varepsilon_p^{\ p}}{d\varepsilon_q^{\ p}} = A\,(M_{cs}-\eta) \qquad [6]$$

in which:

$$A_c = 9/(9 - 2M_{PT,c}\eta + 3M_{PT,c}) \quad \text{in TC} \qquad [7a]$$
$$A_e = 9/(9 - 2M_{PT,e}\eta - 3M_{PT,e}) \quad \text{in TE} \qquad [7a]$$

and $M_{PT,c}$ and $M_{PT,e}$, are obtained using the following relationships:

$$\sin\varphi_{PT,c} = \sin\varphi_{cs} + k_{PT}\,\psi \quad \text{for TC} \qquad [8a]$$

$$\sin\varphi_{PT,e} = \sin\varphi cs + a_{PT} + k_{PT}\,\psi \quad \text{for TE} \qquad [8b]$$

Hardening during shearing is determined from:

$$\frac{\partial p_c}{\partial \varepsilon_q^{\ p}} = \frac{hG}{\left(p_f - p_c\right)_{ini}}(p_f - p_c) \qquad [9]$$

in which h is a non-dimensional material parameter related to soil stiffness during shearing, G is the elastic shear modulus, and $(pf - pc)_{ini}$ is the initial value of $(p_f - p_c)$ at the end of consolidation and prior to shearing. The value of p_f is obtained by substituting the current M_f for η in Equation [1]. Elastic moduli are defined as follows:

$$G = G_r\,\frac{(2.973-e)^2}{1+e}(p/p_a)^{1/2} \qquad [10a]$$

$$K = K_r\,\frac{(2.973-e)^2}{1+e}(p/p_a)^{1/2} \qquad [10b]$$

in which G_r and K_r are reference values that depend on the units used and may be obtained from the elastic moduli corresponding to the atmospheric pressure p_a.