Total versus Differential Settlement of Deep Foundations

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ABSTRACT
This paper investigates the probabilistic nature of differential settlement between two identical piles founded in a spatially variable linearly elastic soil. A theoretical model is developed, and validated by simulation, which is then used to calculate the probability of excessive differential settlement. The theoretical model can be employed in the design of individual piles to avoid excessive differential settlements.

RÉSUMÉ
Cet article étudie la nature probabiliste de tassement différentiel entre deux pieux identiques foncés dans un sol élastique linéaire spatialement variable. Un modèle théorique est développé et validé par simulation et celui-ci est utilisé pour calculer la probabilité de tassement différentiel excessif. Ce modèle théorique est applicable lors de la conception de pieux individuels afin d'éviter un tassement différentiel excessif.

1 INTRODUCTION

Geotechnical foundations are often governed by serviceability limit states (SLS), relating to settlement, rather than by ultimate limit states (ULS), which relate to safety. Most modern geotechnical design codes state that the serviceability limit state can be avoided by designing each foundation to settle by no more than a specified maximum tolerable settlement, δ_{max}. However, in the case of foundations, it is usually differential settlements which govern the serviceability of the supported structure. For example, if all of the foundations of a supported structure settle equally, but excessively, then the approaches to the structure will have to be modified, but the structure itself will not suffer from either a loss of serviceability nor from a loss of safety.

With probability 1, individual foundations will not settle equally and the differential settlement between foundations can lead to loss of serviceability and even catastrophic ultimate limit state failure in the supported structure. So the question is, how should differential settlement between foundations be properly accounted for in the foundation design process?

Although the settlement of deep foundation is not generally a concern if the piles are driven to refusal, settlement can become a design issue if no stiff substratum is encountered. As a result, this paper will concentrate attention on piles which are not end-bearing, that is on piles whose settlement resistance is derived from skin friction and/or adhesion with the surrounding soil.

Design code provisions should be kept as simple as possible, while still achieving a target reliability with respect to both serviceability and ultimate limit states. This means that design codes should retain their maximum settlement requirements but the specified maximum settlement should be reviewed to reasonably ensure that differential settlements do not result in achieving either serviceability or ultimate limit states in the supported structure.

This paper investigates how the maximum settlement specified in a design code for an individual foundation relates to the distribution of the differential settlement between two foundations, as a function of the ground statistics and the distance between the two foundations. Figure 1 illustrates the settlement of two piles founded in a spatially variable ground. The results of this paper can be used to propose design code requirements on maximum settlements for individual foundations which aim to achieve target reliabilities against excessive differential settlements between pairs of foundations.

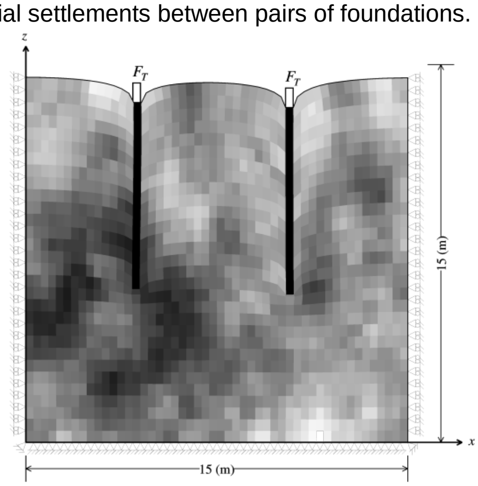

Figure 1. Slice through a random finite element method (RFEM) mesh of a ground supporting two piles.

In this paper, the settlement of a pair of floating piles founded in a three-dimensional spatially random soil mass, each supporting a vertical load, F_T, is studied using the finite random element method (RFEM, Fenton and Griffiths, 2008).

A probabilistic model for differential settlement is presented, which is then validated via Monte Carlo simulation. The results can be used in design provisions for piles that avoid excessive differential settlement.

The paper is organized as follows: In Section 2, a finite element model is presented for a pair of floating piles founded in a three-dimensional spatially random soil mass, each supporting a vertical load. A theoretical approach to estimating the distribution of differential pile settlement is developed in Section 3, and the approach is validated via simulation in Section 4. Conclusions and proposed future work is presented in Section 5.

2 FINITE ELEMENT MODEL

The random settlement of a single pile, which was studied in depth by Naghibi et al. (2014b), is highly dependent on the random elastic modulus field of the surrounding soil, as well as on the pile geometry. In addition, when settlement of pile groups is of interest, mechanical interaction between the piles plays an important role in both total settlement and in differential settlement.

Consider two neighboring piles of identical geometry, supporting loads F_{T_1} and F_{T_2} and separated by distance s, as depicted in Figure 2. For generality in developing the theory, the pile loads will be considered to be random and possibly correlated (although in the validation and the calibration of the mechanical interaction factor, it is assumed that $F_{T_1} = F_{T_2} = F_T = \mu_T$, i.e. non-random). If δ_1' is the settlement of a vertically loaded individual pile without any neighboring piles, and δ_1 is the overall settlement of the pile due to its loading and due to settlement of a neighboring pile, δ_2', then

$$\delta_1 = \delta_1' + \eta\delta_2' \qquad [1]$$

where η is the mechanical interaction factor between the two piles, which is a function of pile spacing and pile length. Rearranging Eq. [1] and solving for η gives

$$\eta = \frac{\delta_1 - \delta_1'}{\delta_2'} \qquad [2]$$

In order to predict η, predictions (or observations) of δ_1, δ_1', and δ_2' are needed. In this paper, these quantities will be found using a linear elastic finite element model of the soil (Smith et al., 2014) with deterministic elastic modulus field ($E = \mu_E$ everywhere, where E is the elastic modulus, and μ_E is its mean). The piles are founded in a three-dimensional linear elastic soil mass modeled using a 50 by 30 by 30 finite element mesh, as illustrated in Figure 1. Eight-node brick elements are used with dimensions 0.3 m by 0.3 m in the x, y (plan) and by 0.5 m in the z (vertical) directions. Within the mesh, piles are modeled as columns of elements having depth H, and hence have a square cross-section with dimension $d = 0.3$ m.

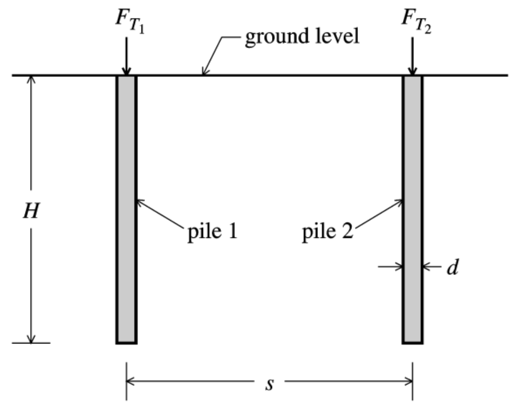

Figure 2. Relative location of two piles

The prediction of η is done for three pile lengths, $H = 2, 4,$ and 8 m, with separation distance, s/d, ranging between 2 and 30, where s is the center-to-center pile spacing and d is the pile diameter. For a particular choice of pile length, H, pile spacing s, applied load, $F_T = 2.16$ MN, pile to soil stiffness ratio $k = E_p / E_s = 700$, soil elastic modulus $E = 30$ MPa, three finite element (FE) analyses are performed; one with pile1 only, one with pile2 only, and one with both pile1 and pile2 separated by distance s. The piles are placed at elements $(50 - s/d)/2$ and $(50 + s/d)/2$, numbered in the x direction. For example, pile1 only case involves the FE analysis of single pile settlement where pile1 is placed at element $(50 - s/d)/2$, while the two pile case involves the FE analysis of two neighboring piles, pile1 and pile2, placed at elements $(50 - s/d)/2$ and $(50 + s/d)/2$, respectively.

Note that the mechanical interaction, η, depends on pile spacing, s, and pile length, H, and is independent of E or F_T. The dependence on Poisson's ratio, ν, is negligible. The piles are placed at least 10 elements away from the boundaries of the FE model, which leads to relative pile settlement error of less than 10% (Naghibi et al., 2014a), so that the influence of boundary conditions on pile settlement is deemed to be negligible.

The resulting sequence of finite element analyses provided predicted values of η for various s/d and pile lengths, H, as shown in Figure 3. It is evident that η increases with increasing pile length, H, and decreases with increasing s/d, as expected. For values of H other than those specified in Figure 3, η is estimated using linear interpolation for $2 \le H \le 8$ and linear extrapolation when H falls outside this range.

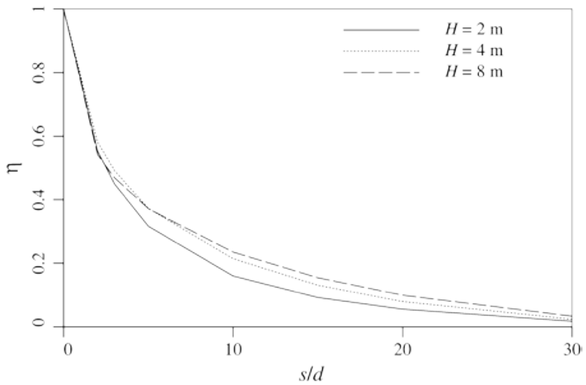

Figure 3. Plot of interaction factor, η, using FE model for $F_T = 2.16$ MN, $k = 700$, $\nu = 0.3$, and $E = 30$ MPa

3 PROBABILISTIC SETTLEMENT MODEL

Attention is now turned to a probabilistic model of pile settlement, where the soil is assumed to be a spatially variable random field. To estimate the pile settlement, it is first assumed that the soil surrounding the pile is perfectly bonded to the pile shaft through friction and/or adhesion. Any displacement of the pile is thus associated with an equivalent displacement of the adjacent soil. Following the classic work of authors such as Poulos and Davis (1980), Randolph and Wroth (1978) and Vesic (1977), the soil is assumed to be linearly elastic, so that this displacement is resisted by a force which is proportional to the soil's elastic modulus and the magnitude of the displacement. Thus, the support provided by the soil to the pile depends on the elastic properties of the surrounding soil.

To design a pile against entering the serviceability limit state, that is, against entering a failure state where the pile's actual settlement exceeds a maximum tolerable settlement, a settlement prediction model is required. If the model is good, then it will provide a good estimate of the mean pile settlement and the in-situ actual pile settlement will be due to natural 'residual' soil variability around the predicted mean. The settlement prediction model is used to determine the pile design such that the predicted mean settlement is some fixed fraction (specified by the load and resistance factors) of the maximum tolerable settlement. If the settlement prediction model is poor, then it also contributes to the variability in the prediction of the actual settlement. This source of variability will be referred to here collectively as the 'degree of site and prediction model understanding', which includes (a) the degree of understanding of the ground properties and geotechnical properties throughout the site, and (b) the accuracy and degree of confidence about the numerical performance prediction model used to estimate the serviceability geotechnical resistances.

It is assumed in this paper that a sufficiently accurate settlement prediction model is used for the pile design, so that model error itself is attributable only to errors in the soil parameters used in the model, that is, to the degree of site understanding. This is probably a reasonable assumption, since if the (possibly non-linear) properties of the soil through which the pile passes, along with the nature of the interface between the pile and the soil, are all well known, then models exist which can provide very good estimates of the mean pile settlement. This paper is not attempting to provide an improved settlement prediction model. In fact a decision about the degree of site and prediction model understanding used in the pile design process is left to the designer. This paper concentrates on the residual settlement variability (around the mean) after the design has been performed. It is assumed that this variability arises from the spatial variability of the soil itself, along with uncertainty in the soil property estimates used in the prediction model.

It is recognized that pile settlement becomes non-linear after about 2% of the pile diameter, and so the elastic modulus mean used in this simulation must be considered to be a secant modulus which approximates the curved nature of the actual pile load-settlement curve. However, the details of the mean settlement predictor used to design a pile and/or to estimate the distribution of differential pile settlement are not important to the subsequent probabilistic analysis (which is relative to the mean), and, of course, the reader is encouraged to use the best settlement prediction available to them. The linear model used in this paper is, however, the best currently available to predict the effects of spatial variability of the soil on the distributions of settlement and differential settlement.

The spatially varying elastic modulus field, which is assumed here to have a constant Poisson's ratio, ν, may be characterized by an equivalent soil elastic modulus, E_g. The equivalent elastic modulus is a spatially uniform value that yields the same settlement as the pile experiences in the actual spatially varying soil (Fenton and Griffiths, 2008). E_g will be assumed here to be the geometric average of the spatially varying elastic modulus field, E, as will be discussed shortly. The elastic modulus is assumed to be lognormally distributed with mean μ_E, standard deviation σ_E, and spatial correlation length, $\theta_{\ln E}$. The lognormal distribution is commonly used to represent non-negative soil properties and means that $\ln E$ is normally distributed with parameters $\mu_{\ln E}$ and $\sigma_{\ln E}$. The distribution parameters of $\ln E$ can be obtained from the mean and standard deviation of E using the following transformations

$$\mu_{\ln E} = \ln(\mu_E) - \frac{1}{2}\sigma^2_{\ln E} \qquad [3]$$
$$\sigma^2_{\ln E} = \ln(1 + v_E^2)$$

where $v_E = \sigma_E / \mu_E$ is the coefficient of variation of E. If the soil's elastic modulus, E, is lognormally distributed, as assumed, then E_g will also be lognormally distributed since geometric averages preserve the lognormal distribution (Fenton and Griffiths, 2008).

The correlation coefficient between the log elastic modulus at two points is defined by a correlation function, $\rho_{\ln E}(\tau)$, in which τ is the distance between the two points. In this study, a simple isotropic exponentially decaying (Markovian) correlation function will be employed, having the form

$$\rho_{\ln E}(\tau) = \exp\left\{-\frac{2|\tau|}{\theta_{\ln E}}\right\} \qquad [4]$$

where τ is the distance between any two points in the field and $\theta_{\ln E}$ is the correlation length (Fenton and Griffiths, 2008).

Since the soil is a spatially variable random field, the pile settlement will also be random. Assuming that the pile settlement is approximated lognormally distributed (as was shown to be reasonable by Naghibi et al., 2014b), then the task is to find the parameters of that distribution and the distribution of the resulting differential settlement. Assuming further that δ_1 and δ_2 are the total settlements of the two piles shown in Figures 1 and 2, then δ_1 and δ_2 are identically and lognormally distributed random variables.

The differential settlement between two piles is defined here to be $\Delta = \delta_1 - \delta_2$. If the elastic modulus field is statistically stationary, as assumed here, then the mean differential settlement, μ_Δ, is zero. The mean absolute differential settlement can be approximated by (if Δ is approximately normally distributed)

$$\mu_{|\Delta|} \simeq \sqrt{\frac{2}{\pi}}\sigma_\Delta \qquad [5]$$

which indicates that the mean of the absolute differential settlement is directly related to the standard deviation of Δ, and hence related to the variability of the elastic moduli surrounding the piles. The approximation in Eq. [5] is exact if Δ is normally distributed (Papoulis, 1991), and, as will be shown shortly, this approximation is in reasonable agreement with simulation based results.

Investigations by Fenton and Griffiths (2002) suggest that the equivalent elastic modulus as seen by a shallow foundation is a geometric average of the soil's elastic modulus under the foundation. Naghibi et al. (2014b) similarly assumes that the equivalent elastic modulus, E_g, as seen by a pile is a geometric average of the soil's elastic modulus over some volume, V_f, surrounding the pile

$$\begin{aligned} E_g &= \exp\left\{\frac{1}{V_f}\int_{V_f} \ln E(\underset{\sim}{x})\,d\underset{\sim}{x}\right\} \\ &= \exp\left\{\frac{1}{B^2C}\int_0^B\int_0^B\int_0^C \ln E(x,y,z)\,dz\,dy\,dx\right\} \end{aligned} \qquad [6]$$

where $E(\underset{\sim}{x}) = E(x,y,z)$ is the elastic modulus of the soil at spatial position *(x, y, z)*. The pile is centered on the volume $V_f = B \times B \times C$ where *C* is measured in the vertical (*z*) direction.

The settlement of a single pile can then be expressed as

$$\delta_i' = \delta_{\det}\left(\frac{\mu_E}{E_g}\right)\left(\frac{F_{T_i}}{\mu_T}\right) \qquad [7]$$

where the subscript *i* is either 1 or 2, and $\delta_{\det}$ is the deterministic settlement of a single pile obtained from a single finite element analysis of the problem using $F_{T_i} = \mu_T$ and $E = \mu_E$ everywhere. Substituting Eq. [7] into Eq. [1] leads to pile settlements, δ_1 and δ_2, as follows

$$\begin{aligned} \delta_1 &= \delta_{\det}\left(\frac{\mu_E}{\mu_T}\right)\left(\frac{F_{T_1}}{E_{g_1}} + \frac{\eta F_{T_2}}{E_{g_2}}\right) \\ \delta_2 &= \delta_{\det}\left(\frac{\mu_E}{\mu_T}\right)\left(\frac{F_{T_2}}{E_{g_2}} + \frac{\eta F_{T_1}}{E_{g_1}}\right) \end{aligned} \qquad [8]$$

The differential settlement, $\Delta = \delta_1 - \delta_2$, between two piles becomes

$$\Delta = \delta_{\det}(1-\eta)\left(\frac{\mu_E}{\mu_T}\right)\left(\frac{F_{T_1}}{E_{g_1}} - \frac{F_{T_2}}{E_{g_2}}\right) \qquad [9]$$

The variance of Δ is therefore

$$\sigma_\Delta^2 = \delta_{\det}^2(1-\eta)^2\left(\frac{\mu_E}{\mu_T}\right)^2 \mathrm{Var}\left[\frac{F_{T_1}}{E_{g_1}} - \frac{F_{T_2}}{E_{g_2}}\right] \qquad [10]$$

where

$$\begin{aligned} &\mathrm{Var}\left[\frac{F_{T_1}}{E_{g_1}} - \frac{F_{T_2}}{E_{g_2}}\right] = \\ &\mathrm{Var}\left[\frac{F_{T_1}}{E_{g_1}}\right] + \mathrm{Var}\left[\frac{F_{T_2}}{E_{g_2}}\right] - 2\mathrm{Cov}\left[\frac{F_{T_1}}{E_{g_1}}, \frac{F_{T_2}}{E_{g_2}}\right] \end{aligned} \qquad [11]$$

Now if $X_i = F_{T_i}/E_{g_i}$, $i = 1,2,$ and F_{T_i} and E_{g_i} are lognormally distributed, then X_i is also lognormally distributed. In this case, $\ln X_i = \ln F_{T_i} - \ln E_{g_i}$ is normally distributed with parameters (Naghibi et al., 2014b)

$$\begin{aligned} \mu_{\ln X_i} &= \mu_{\ln F_T} - \mu_{\ln E_g} = \mu_{\ln F_T} - \mu_{\ln E} \\ &= \ln(\mu_T) - \frac{1}{2}\sigma_{\ln F_T}^2 - \ln(\mu_E) + \frac{1}{2}\sigma_{\ln E}^2 \\ \sigma_{\ln X_i}^2 &= \sigma_{\ln F_T}^2 + \sigma_{\ln E_g}^2 = \sigma_{\ln F_T}^2 + \sigma_{\ln E}^2\gamma_f \end{aligned} \qquad [12]$$

assuming that F_{T_i} and E_{g_i} are independent, and where γ_f is the variance reduction due to averaging $\ln E$ over the three-dimensional volume $V_f = B \times B \times C$ surrounding the piles. In detail,

$$\gamma_f = \frac{1}{V_f^2}\int_0^{V_f}\int_0^{V_f} \rho_{\ln E}(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)\,d\underset{\sim}{x}_1\,d\underset{\sim}{x}_2 \qquad [13]$$

where $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ are two spatial positions within V_f. Note that γ_f is essentially just the average correlation coefficient between all points within the volume V_f .

The distribution parameters of X_i can be obtained from the mean and standard deviation of $\ln X_i$ using the following transformations

$$\mu_{X_i} = \exp\left\{\mu_{\ln X_i} + \frac{1}{2}\sigma^2_{\ln X_i}\right\}$$
$$\sigma^2_{X_i} = \mu^2_{X_i}\left(e^{\sigma^2_{\ln X_i}} - 1\right) \qquad [14]$$

Using Eq.'s [12] in Eq.'s [14] results in

$$\begin{aligned}\mu_{X_i} &= \mathrm{E}\left[\frac{F_{T_i}}{E_{g_i}}\right] \\ &= \exp\left\{\ln(\mu_{\mathrm{T}}) - \frac{1}{2}\sigma^2_{\ln F_T} - \ln(\mu_E) + \frac{1}{2}\sigma^2_{\ln E} + \frac{1}{2}\sigma^2_{\ln E}\gamma_f\right\} \\ &= \frac{\mu_{\mathrm{T}}}{\mu_E}\sqrt{\left(1+v_E^2\right)^{\gamma_f+1}} \\ \sigma^2_{X_i} &= \mathrm{Var}\left[\frac{F_{T_i}}{E_{g_i}}\right] \\ &= \left(\mu_T^2(1+v_E^2)^{\gamma_f+1} / \mu_E^2\right)\left(\exp\left\{\sigma^2_{\ln F_T} + \sigma^2_{\ln E}\gamma_f\right\} - 1\right) \\ &= \frac{\mu_T^2}{\mu_E^2}(1+v_E^2)^{\gamma_f+1}\left((1+v_T^2)(1+v_E^2)^{\gamma_f} - 1\right)\end{aligned} \qquad [15]$$

where $v_E = \sigma_E / \mu_E$ is the coefficient of variation of the elastic modulus field, E .

The covariance between the two lognormal random variables $X_1 = F_{T_1} / E_{g_1}$ and $X_2 = F_{T_2} / E_{g_2}$ can be computed as

$$\mathrm{Cov}\left[X_1, X_2\right] = \mathrm{Cov}\left[\frac{F_{T_1}}{E_{g_1}}, \frac{F_{T_2}}{E_{g_2}}\right] = \sigma^2_{X_i}\rho_X \qquad [16]$$

where $\sigma^2_{X_i}$ is given by Eq. [15], and the correlation coefficient, ρ_X , comes from the transformation (Fenton and Griffiths, 2008)

$$\rho_X = \frac{\exp\left\{\mathrm{Cov}\left[\ln X_1, \ln X_2\right]\right\} - 1}{\exp\left\{\sigma^2_{\ln X_i}\right\} - 1} \qquad [17]$$

and $\sigma^2_{\ln X_i}$ is defined by Eq. [12]. In addition

$$\begin{aligned}&\mathrm{Cov}\left[\ln X_1, \ln X_2\right] \\ &= \mathrm{Cov}\left[\ln F_{T_1} - \ln E_{g_1}, \ln F_{T_2} - \ln E_{g_2}\right] \\ &= \mathrm{Cov}\left[\ln F_{T_1}, \ln F_{T_2}\right] + \mathrm{Cov}\left[\ln E_{g_1}, \ln E_{g_2}\right] \\ &\simeq \sigma^2_{\ln F_T}\rho_{\ln F_T} + \sigma^2_{\ln E}\gamma_{ff}\end{aligned} \qquad [18]$$

again assuming F_T and E_g are independent. The correlation $\rho_{\ln F_T}$ is given by

$$\rho_{\ln F_T} = \frac{\ln\left(1+\rho_{F_T}v_T^2\right)}{\ln\left(1+v_T^2\right)} = \frac{\ln\left(1+\rho_{F_T}v_T^2\right)}{\sigma^2_{\ln F_T}} \qquad [19]$$

where ρ_{F_T} is the correlation between loads F_{T_1} and F_{T_2} . The term γ_{ff} is the average correlation coefficient between two log-elastic modulus fields of sizes $V_f = B \times B \times C$ surrounding the two piles, which are separated by distance s (see Figure 2 for s). In detail, γ_{ff} is given by

$$\gamma_{ff} = \frac{1}{V_f^2}\int_0^{V_f}\int_0^{V_f}\rho_{\ln E}\left(\sqrt{s^2 + (\underset{\sim}{x}_1 - \underset{\sim}{x}_2)}\right)^2 d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2 \qquad [20]$$

where $\rho_{\ln E}$ is given by Eq. [4].

Employing Eq's. [12] and [18] in Eq. [17] leads to

$$\begin{aligned}\rho_X &= \frac{\exp\left\{\sigma^2_{\ln F_T}\rho_{\ln F_T} + \sigma^2_{\ln E}\gamma_{ff}\right\} - 1}{\exp\left\{\sigma^2_{\ln F_T} + \sigma^2_{\ln E}\gamma_f\right\} - 1} \\ &= \frac{\left(1+\rho_{F_T}v_T^2\right)\left(1+v_E^2\right)^{\gamma_{ff}} - 1}{\left(1+v_T^2\right)\left(1+v_E^2\right)^{\gamma_f} - 1}\end{aligned} \qquad [21]$$

and using Eq.[15] and [21] in Eq. [16] results in

$$\begin{aligned}&\mathrm{Cov}\left[X_1, X_2\right] \\ &= \frac{\mu_T^2}{\mu_E^2}\left(1+v_E^2\right)^{\gamma_f+1}\left(\left(1+\rho_{F_T}v_T^2\right)\left(1+v_E^2\right)^{\gamma_{ff}} - 1\right)\end{aligned} \qquad [22]$$

Finally substituting Eq's. [15] and [22] into Eq. [11] gives

$$\begin{aligned}&\mathrm{Var}\left[\frac{F_{T_1}}{E_{g_1}} - \frac{F_{T_2}}{E_{g_2}}\right] = 2\frac{\mu_T^2}{\mu_E^2}\left(1+v_E^2\right)^{\gamma_f+1} \\ &\left[\left(1+v_T^2\right)\left(1+v_E^2\right)^{\gamma_f} - \left(1+\rho_{F_T}v_T^2\right)\left(1+v_E^2\right)^{\gamma_{ff}}\right]\end{aligned} \qquad [23]$$

from which the variance of differential settlement, σ^2_Δ, becomes

$$\begin{aligned}&\sigma^2_\Delta = 2\delta^2_{\mathrm{det}}(1-\eta)^2\left(1+v_E^2\right)^{\gamma_f+1} \times \\ &\left[\left(1+v_T^2\right)\left(1+v_E^2\right)^{\gamma_f} - \left(1+\rho_{F_T}v_T^2\right)\left(1+v_E^2\right)^{\gamma_{ff}}\right]\end{aligned} \qquad [24]$$

If $\rho_{F_T} = 1$, so that the loads F_{T_1} and F_{T_2} are the same, then the above equation simplifies to,

$$\sigma_\Delta^2 = 2\delta_{\det}^2 (1-\eta)^2 \left(1+v_E^2\right)^{\gamma_f+1}\left(1+v_T^2\right)\times \left[\left(1+v_E^2\right)^{\gamma_f} - \left(1+v_E^2\right)^{\gamma_{ff}}\right] \quad [25]$$

For $s \to \infty$, pile settlements δ_1 and δ_2 becomes independent, and hence both η and γ_{ff} in Eq. [25] become zero. In this case, Eq. [25] reduces to

$$\sigma_\Delta^2 = 2\delta_{\det}^2 \left(1+v_E^2\right)^{\gamma_f+1}\left(1+v_T^2\right)\left[\left(1+v_E^2\right)^{\gamma_f} - 1\right] \quad [26]$$

Assuming that the normal distribution is a reasonable approximate distribution of the differential settlement between two piles, then the probability that the differential settlement exceeds $\Delta_{\max}$ is

$$\begin{aligned} p_f &= \mathrm{P}\left[|\Delta| > \Delta_{\max}\right] = \mathrm{P}\left[\Delta < -\Delta_{\max} \cup \Delta > \Delta_{\max}\right] \\ &= 2\mathrm{P}\left[\Delta < -\Delta_{\max}\right] = 2\Phi\left(\frac{\Delta_{\max} - \mu_\Delta}{\sigma_\Delta}\right) \\ &= 2\Phi\left(\frac{\Delta_{\max}}{\sigma_\Delta}\right) = \Phi(-\beta) \end{aligned} \quad [27]$$

since $\mu_\Delta = 0$, and where σ_Δ is calculated via Eq. [24]. The probability of failure, p_f, can also be expressed in terms of the reliability index, as shown in Eq. [27], where Φ is the standard normal cumulative distribution function. That is, the reliability index corresponding to a particular value of p_f can be obtained by inverting Eq. [27], $\beta = -\Phi^{-1}(p_f)$.

4 VALIDATION OF THEORY VIA MONTE CARLO SIMULATION

In this section, the predicted parameters of the distribution of differential settlement, Δ, are compared to Monte Carlo simulation results in order to assess the accuracy of the theory developed in the previous section. The particular case considered in this validation study is detailed in Table 1.

Realizations of differential settlement of two piles are obtained using the random finite element method (RFEM) (Fenton and Griffiths, 2008).

Three load cases (F_T) are considered for this analysis, as listed in Table 1 and it is assumed that the pile loads are equal and non-random. For each load case, a design pile length is determined as follows to achieve a target maximum settlement, $\delta_{\max} = 0.025$ m (Naghibi et al., 2014a)

$$H = d\left[\left(\frac{1}{\left(\delta_{\max}\varphi_{gs}\mu_E d / F_T\right) - a_0}\right)^{1/a_2} - a_1\right] \quad [28]$$

Table 1. Input parameters used in the validation of theory

Parameters	Values Considered
d	0.3 m
E_p	21 Gpa
F_T	1.46, 2.16, 3.16 MN
μ_E	30 Mpa
v_E	0.1, 0.3, 0.5
Poisson's ratio, ν	0.3
$\theta_{\ln E}$	0.01, 0.1, 0.5, 1.0, 5.0, 10.0 m
s	2*d*, 3*d*, 5*d*, 10*d*, 15*d*, 20*d*, 30*d*
n_{sim}	2000
B	2 m
C	$2H$

where φ_{gs} is the geotechnical resistance factor, accounting for uncertainty in geotechnical resistance. In this validation study, φ_{gs} is taken to be 1.0. The pile to soil stiffness ratio is assumed to be $k = 700$, from which $a_0 = 0.029$, $a_1 = 2.44$, $a_2 = 0.939$ are obtained using the regression developed by Naghibi et al. (2014a). The resulting design pile lengths are $H = 2, 4,$ and 8 m for the three load cases $F_T = 1.46, 2.16,$ and 3.16 MN respectively.

The mechanical interaction factor, η, is obtained for each pile length (load case) and each pile spacing, s, listed in Table 2, using Figure 3.

The soil volume surrounding the pile, $V_f = B \times B \times C$, for use in the geometric average given by Eq. [6], was selected by trial and error (see Naghibi et al., 2014b) and the (approximately) best averaging volume was found to occur when $B = 2$ m, and $C = 2H$. These choices led to the best agreement between theory and simulation with respect to settlement exceedance probabilities for a single pile.

Figures 4 and 5 illustrate the comparison between the theory and simulation-based estimates of $\mu_{|\Delta|}$ and σ_Δ. The theoretical estimates were obtained using Eq's. [5] and [24] for $\rho_{F_T} = 1$ (identical loads), and $v_T = 0$ (non-random load). Figure 4 demonstrates that the theory underestimates $\mu_{|\Delta|}$ when $\mu_{|\Delta|}$ is small. Although not seen in the figure, the underestimation occurs most strongly for longer piles and smaller values of v_E and $\theta_{\ln E}$.

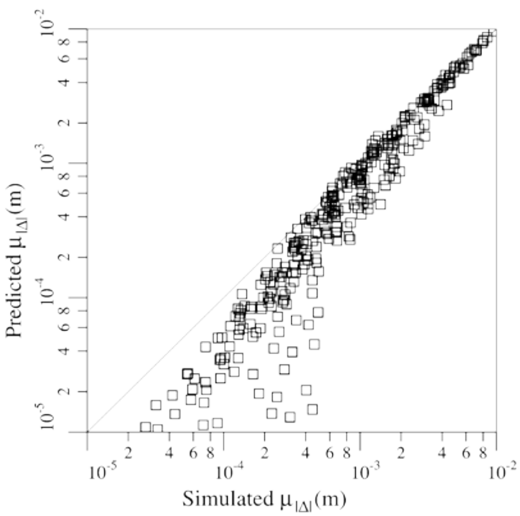

Figure 4. Predicted, obtained via Eq. [5], versus simulated mean absolute differential settlement, $\mu_{|\Delta|}$, for all cases listed in Table 1

The discrepancies between simulation and theory seen in Figures 4 and 5 arise mainly due to approximations made in the theory. For instance, the theory assumes that Δ is normally distributed – this is not a good assumption when $\mu_{|\Delta|}$ is small, which is where the errors become more pronounced. Note that the errors are actually quite small in absolute value and thus of not great importance. For example, when the simulated $\mu_{|\Delta|}$ is less than 0.5 mm, the predicted $\mu_{|\Delta|}$ is often less than about 0.1 mm. In either case, the differential settlement is negligible.

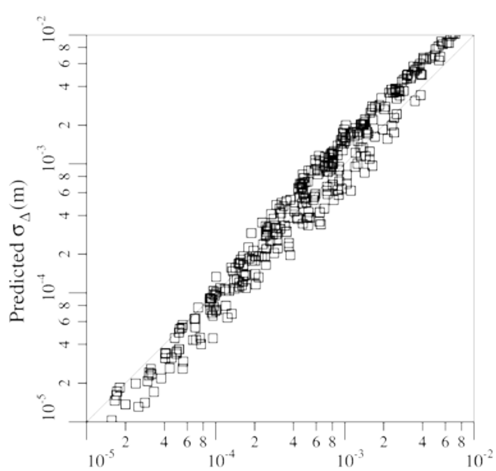

Figure 5. Predicted, obtained via Eq. [24], versus simulated standard deviation differential settlement, σ_Δ, for all cases listed in Table 1

The probability that the differential settlement exceeds $\Delta_{\max}$, as predicted by Eq. [27], is compared to simulation in Figure 6 for three possible maximum acceptable differential settlement to pile spacing ratio values, $\Delta_{\max}/s = 1/200, 1/500,$ and 1/1000. These serviceability gradient limitations are as specified in the Canadian Foundation Engineering Manual (CFEM) (Canadian Geotechnical Society, 2006).

It is evident from Figure 6 that the theory sometimes significantly underestimates $\mathrm{P}\left[|\Delta| > \Delta_{\max}\right]$, which is unconservative. Although not shown in Figure 6, the disagreement is worst for smaller pile spacings, s.

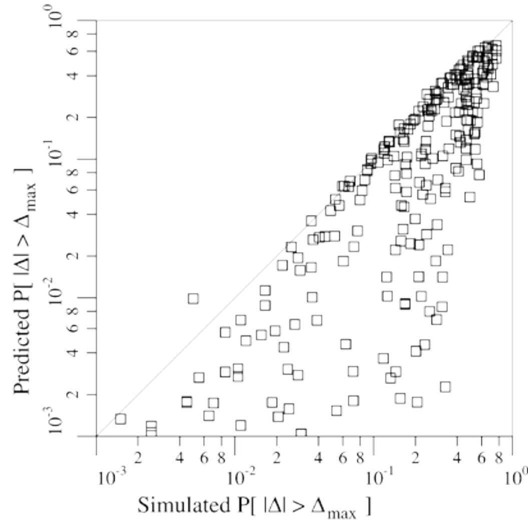

Figure 6. Predicted, via Eq. [27], versus simulated $\mathrm{P}\left[|\Delta| > \Delta_{\max}\right]$ for all cases listed in Table 1

It is believed that the discrepancy between theory and simulation in Figure 6 is due to the covariance between the piles being essentially overestimated in the theory by including both a statistical covariance component (γ_{ff}) at the same time as a mechanical interaction term (η). The overestimation in the 'equivalent' covariance between the piles reduces the theoretically predicted mean differential settlement, as seen in Figure 4, and thus significantly reduces the theoretical probability of excessive settlement, as seen in Figure 6. This discrepancy can be largely solved by introducing an empirical correction to the theory, which was found by trial-and-error. If the value of η is replaced by -0.5η for short piles (e.g., $H < 3$ m), by 0 for medium length piles (e.g., $3 \le H \le 6$ m), and by 0.5η for longer piles (e.g., $H > 6$ m), then the agreement between theoretical and simulated exceedance probabilities is significantly improved, as shown in Figure 7.

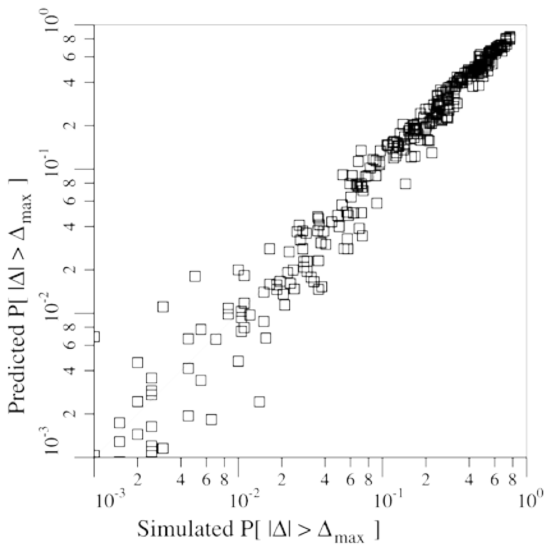

Figure 7. Predicted, obtained via Eq. [27], versus simulated $\mathrm{P}\left[\left|\Delta\right| > \Delta_{\max}\right]$, corrected by replacing η with -0.5η for $H < 3$ m, with zero for $3 \le H \le 6$ m, and with 0.5η for $H > 6$ m.

What this empirical correction is essentially doing is reducing the covariance between the piles – the amount of covariance reduction is greatest for shorter piles, where most of the errors were seen, since the statistical covariance portion is relatively higher for shorter piles. The application of an empirical correction raises the question as to why the statistical covariance isn't reduced simply by reducing the volume $V_f = B \times B \times C$, rather than by reducing the mechanical component? The reason is that the choice in V_f led to a good prediction of the distribution of the settlement of an individual pile, and so it was felt that its size was an appropriate measure of the zone around the pile influencing the total settlement. The problem really is that the mechanical interaction factor was determined from a deterministic (non-random elastic modulus field). The actual mechanical interaction factor in a spatially random elastic modulus field is unknown and not easy to specify probabilistically. Preliminary results using trials having identical random field realizations indicate that the random mechanical interaction factor is always lower than the deterministic mechanical interaction factor. However, it was felt that the determination of the distribution of the actual η was beyond the scope of this paper, and perhaps not worth the effort, since the empirical correction suggested above seems to work so well.

With this empirical correction, the agreement between theory and simulation is considered very good. Thus, the theory is believed to be reliable enough to assist in design recommendations (to be published by authors shortly). In other words, the normal distribution, along with an empirically corrected standard deviation, can be used as a reasonable approximation to the distribution of the differential settlement between two piles.

5 CONCLUSIONS

The differential settlement between two piles is studied and a theoretical model with an empirical correction is developed, which is then validated by simulation. The theoretical model can be used to estimate the probability of excessive differential settlement and hence to provide design recommendations. The relationship between the target maximum settlement recommended in design codes and the distribution of the differential settlement between two piles was investigated.

The differential settlement model presented here is a function of the load and the ground stiffness distributions, along with the distance, correlation coefficient (in both loads and ground parameters), and mechanical interaction between the piles. The local averages used around the piles gave very good agreement between predicted and simulated exceedance probabilities for total settlement in the study by Naghibi et al. (2014b). However, using the same local averages in this paper overemphasized the correlation between piles. To compensate, an empirical adjustment factor was introduced. The resulting probabilistic model is quite general and the agreement between the model and differential settlement simulation results was deemed to be very good (see Figure 7).

Note that this study is based on variability in the ground and not on variability in the loads. That is, $v_T = 0$ was used in the validation of theory via simulation and the loads applied to the two piles were assumed equal. The actual joint load distribution is dependent on the stiffness of the supported structure, amongst other things, and the stiffness of the supported structure also influences the maximum allowable differential settlement. To properly model the joint load distribution, and the maximum tolerable differential settlement, a model of the supported structure would be required, which was beyond the scope of this paper. The assumptions made here essentially correspond to that of a very stiff supported structure (e.g., a pile cap). If the structure is actually quite flexible, one would expect increased differential settlement, but at the same time one would expect additional tolerance for differential movement. It is felt that the results presented here are basically applicable regardless of the structural model. Nevertheless, research is ongoing into the effect that structural stiffness and load transfer mechanisms have on the SLS design of individual foundations against excessive differential movement.

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