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Role of air-entry value and choice of SWCC in the prediction of the unsaturated permeability

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ABSTRACT

The permeability function is commonly estimated by integrating along the soil-water characteristic curve (SWCC) based on a particular integral formula. The Fredlund, Xing and Huang (1994) permeability function is a commonly used estimation technique. It has become common practice to start the integration procedure from a value near zero rather than the originally specified air-entry value (AEV). This paper undertakes a study on the effect of the lower limit of integration on the estimation of the permeability function. A mathematical algorithm is also proposed for the calculation of the AEV for integration purposes. The results reveal that the relative coefficient of permeability can be significantly under-estimated if the lower limit of integration is smaller than the AEV. The recommendation is that the AEV always be used as the lower limit of integration when using the Fredlund et al (1994) permeability estimation equation.

RÉSUMÉ

La fonction de perméabilité est fréquemment estimée en intégrant la courbe de rétention d'eau (SWCC) avec une formule intégrante particulière. La fonction de perméabilité de Fredlund, Xing et Huang (1994) est une méthode d'estimation fréquemment utilisée. C'est devenu une pratique courante de commencer la procédure d'intégration avec une valeur proche du zéro, plutôt qu'avec la valeur d'entrée d'air (AEV) comme indiqué originalement. Cet article étudie l'influence de la limite inférieure d'intégration sur l'estimation de la fonction de perméabilité. Un algorithme mathématique est aussi proposé pour calculer la AEV pour des fins d'intégration. Les résultats révèlent que le coefficient de perméabilité relative peut être significativement sous-estimé si la limite inférieure d'intégration est plus petite que la AEV. L'étude montre que la AEV doit toujours être utilisée comme limite inférieure d'intégration lorsque l'équation d'estimation de la perméabilité de Fredlund et coll. (1994) est utilisée.

1 INTRODUCTION

The unsaturated coefficient of permeability function is an important soil property function used in the numerical modeling of saturated-unsaturated soil systems. Direct measurement of the unsaturated permeability function is costly, technically-demanding, and time-consuming. Therefore, considerable research has been directed towards the estimation of the unsaturated coefficient of permeability function. There are four categories of models used for the estimation of unsaturated coefficient of permeability functions (Fredlund et al. 2012), namely, i.) empirical models, ii.) statistical models, iii.) correlation models and iv.) regression models. Empirical models and statistical models appear to be most extensively used in geotechnical engineering.

Empirical models make use of the similar character of the soil-water characteristic curve, (SWCC), and the permeability function in order to estimate the unsaturated coefficient of permeability function. The Brooks and Corey (1964) equation is one example of an empirical model. Statistical models make use of the fact that the permeability function and the soil-water characteristic curve are mainly controlled by the pore-size distribution of the soil. Consequently, the permeability function was developed based on the interpretation and application of the SWCC. Childs and Collis-George (1950), Burdine (1953) and Mualem (1976) are three commonly used integral formulas of relative permeability based on different physical models.

The van Genuchten-Burdine (1980) equation, the van Genuchten-Mualem (1980) equation, and the Fredlund, Xing and Huang (1994) permeability function are three permeability functions for unsaturated soils commonly used in geotechnical engineering. These three unsaturated coefficient of permeability functions are developed by introducing various mathematical equations for the SWCC into different integral formulas based on different physical models. The unsaturated soil permeability function is obtained by combining the saturated coefficient of permeability and the relative coefficient of permeability. The Fredlund, Xing and Huang (1994) permeability function is an integral solution for the permeability equation, obtained by introducing the

Fredlund and Xing (1994) SWCC equation into the Childs and Collis-George (1950) integral formula. The resulting permeability function has the advantage that the integral permeability function retains the independence of the SWCC fitting variables when estimating the permeability function. On the other hand, the van Genuchten permeability function has a closed-form and is simpler to use in engineering practice.

The original relative permeability theory published by Fredlund et al. (1994) specified the air-entry value, ψ_{aev}, as the lower limit of integration. However, implementations in engineering practice appear to have used other values between zero and ψ_{aev} as the starting point of integration when calculating the relative coefficient of permeability. It doesn't appear that any study has been undertaken to assess whether the choice for the lower limit of integration influences the calculation of the Fredlund, Xing and Huang (1994) permeability function. In addition, the importance of using the degree of saturation SWCC (*S*-SWCC) for calculating the permeability function has not been clearly emphasized in the research literature.

This paper lays out an empirical procedure for the determination of the air-entry value and investigates the error caused by using various values for the lower limit of integration. The effect of the choice of SWCC on the estimation of the relative permeability function is also studied.

2 DETERMINATION OF THE AIR-ENTRY VALUE (AEV) FROM THE DEGREE OF SATURATION SWCC, (*S*-SWCC)

2.1 Different forms of soil-water characteristic curve (SWCCs)

The SWCC for a soil is defined as the relationship between the water content and soil suction (Williams 1982), and is commonly used as the basis for the estimation of unsaturated soil properties (e.g., the permeability function for an unsaturated soil). Different designations for the amount of water in the soil generate different forms of SWCC. The designations for these SWCCs can be referred to as the: gravimetric water content SWCC, volumetric water content SWCC, instantaneous volumetric water content SWCC, and degree of saturation SWCC. The volumetric water content is the water content with the volume of water referenced to the original total volume of the soil specimen. The instantaneous volumetric water content is the water content with the volume of water referenced to the instantaneous total volume of the soil specimen. Each form of the SWCC provides similar information to the geotechnical engineer if the soil does not undergo volume change as soil suction is increased (as shown in Figure 1).

When soil undergoes volume change, as is the case for soft clays and slurry soils, the gravimetric water content SWCC, instantaneous volumetric water content SWCC and degree of saturation SWCC are distinctly different from one another (as shown in Figure 2). Volumetric water content SWCC is not of significance when soil undergoes high volume change. Conventional permeability functions (e.g., Fredlund et al. (1994) equation; van Genuchten-Burdine (1980) equation; van Genuchten-Mualem (1980) equation) have been proposed based on the assumption that there is little or no volume change as the soil dries. The volumetric water content SWCC is no longer appropriate for the estimation of the relative permeability function when soil undergoes volume change. It is important to know that the relative coefficient of permeability function, as well as the air-entry value must be estimated from degree of saturation SWCC (Fredlund et al. 2011). This paper uses the degree of saturation SWCC to calculate the appropriate estimation of the relative permeability function.

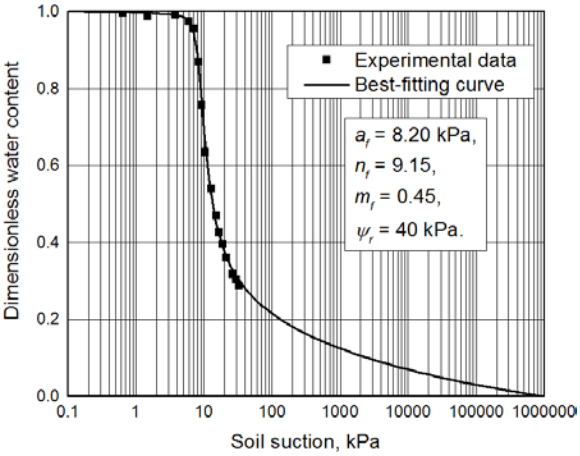

Figure 1. SWCC experimental data for GE3 and its best-fitting curve. (Data from Brooks and Corey 1964)

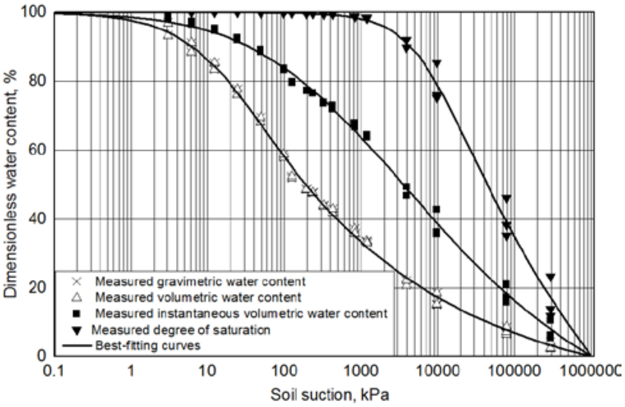

Figure 2. SWCC experimental data for Regina clay and its best-fitting curves. (Data from Fredlund 1964)

Various forms of mathematical equations have been suggested to characterize the SWCC. The equation proposed by Fredlund and Xing (1994) has been shown to have sufficient flexibility to best-fit laboratory data reasonably well over the entire soil suction range from near zero to 106 kPa provided the material behaves in a mono-modal manner. The form of the Fredlund and Xing (1994) equation written in terms of degree of saturation, (i.e., S-SWCC) is shown in Eq. 1.

$$S(\psi) = \frac{S_0\left(1-\ln\left(1+\psi/\psi_r\right)/\ln\left(1+10^6/\psi_r\right)\right)}{\left(\ln\left(\exp(1)+\left(\psi/a_f\right)^{n_f}\right)\right)^{m_f}} \qquad [1]$$

where,
ψ = soil suction;
$S(\psi)$ = degree of saturation at a soil suction of ψ;
S_0 = initial degree of saturation at zero soil suction; and
a_f, n_f, m_f, ψ_r = four best-fitting parameters controlling the shape of the SWCC.

2.2 Mathematical algorithm for the empirical determination of the air-entry value (AEV)

The air-entry value, (AEV), of the soil is the suction at which air begins to enter the largest pores in the soil (Fredlund and Xing 1994). Vanapalli et al. (1998) proposed an empirical, graphical construction technique to estimate the air-entry value from the SWCC. The air-entry value must be determined from the degree of saturation SWCC (Fredlund et al. 2011).

A mathematical algorithm is proposed in this paper for the determination of the AEV based on the graphical construction method suggested by Vanapalli et al., (1998). The following steps are outlined with respect to the analysis for the AEV.

Step 1. Find the best-fitting curve for the degree of saturation SWCC using the Fredlund and Xing (1994) equation (Figure 3).

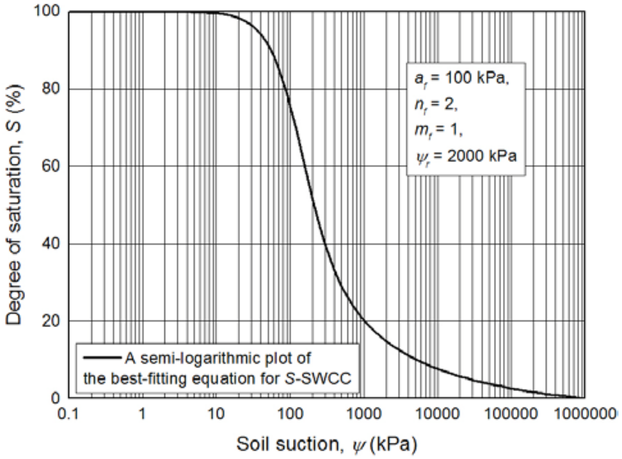

Figure 3. *S*-SWCC for a hypothetical soil plotted using semi-logarithmic coordinate.

Step 2. Through use of a variable substitution technique, the Fredlund-Xing (1994) best-fitting equation can be transformed into a substitution equation (i.e., Eq. 2). The substitution equation describes the relationship between the degree of saturation and the logarithm of soil suction to base 10 (Figure 4). The shape of the curve for the substitution equation plotted using arithmetic coordinates is the same as the shape of the curve for the best-fitting equation plotted using a semi-logarithmic coordinate system. The arithmetic plot of the substitution equation has the same inflection point as the semi-logarithmic plot of the best-fitting equation.

$$SS(\xi) = \frac{S_0\left(1-\ln\left(1+10^{\xi}/\psi_r\right)/\ln\left(1+10^6/\psi_r\right)\right)}{\left(\ln\left(\exp(1)+\left(10^{\xi}/a_f\right)^{n_f}\right)\right)^{m_f}} \qquad [2]$$

where, $\xi = \mathrm{Log}_{10}(\psi)$; $SS(\xi)$ = the degree of saturation at a soil suction of ψ; ψ = soil suction.

Step 3. Determine the point of maximum slope (or the inflection point) on the arithmetic plot of the substitution equation. The point of maximum slope is also a point of zero curvature. Therefore, the second derivative of Eq. 2 can be set equal to zero as shown in Eq. 3.

$$\frac{\mathrm{d}^2 SS(\xi)}{\mathrm{d}\xi^2} = 0 \qquad [3]$$

Solving Eq. 3 for the ξ value of zero curvature point and substituting the ξ value into Eq. 2 yields the corresponding term, $SS(\xi)$. The determined point of zero curvature has coordinates (ξ_i, $SS(\xi_i)$) (Figure 4).

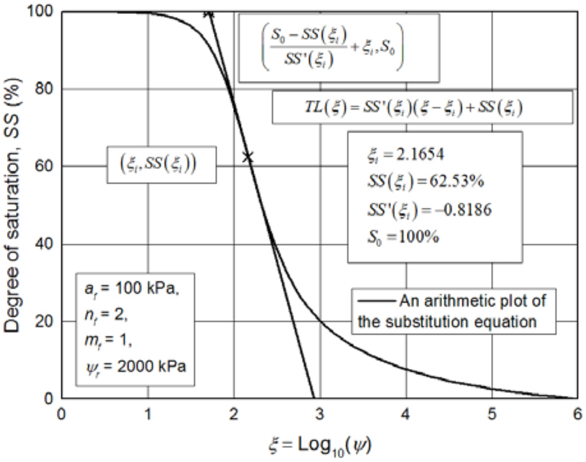

Figure 4. The arithmetical plot of the substitution equation.

Step 4. Draw a line tangent to the curve through the point of maximum slope (Figure 4). The point of maximum slope is (ξ_i, $SS(\xi_i)$) and the maximum slope is $SS'(\xi_i)$. The equation for the tangent line is as shown in Eq. 4.

$$TL(\xi) = SS'(\xi_i)(\xi - \xi_i) + SS(\xi_i) \qquad [4]$$

where, $TL(\xi)$ represents the function of the tangent line.

Step 5. Draw a horizontal line through the maximum degree of saturation. The intersection of the two lines indicates the air-entry value (Figure 4). The horizontal line is given by Eq. 5.

$$HL(\xi) = S_0 \qquad [5]$$

where, $HL(\xi)$ represents the function of the horizontal line.

The intersection point can be obtained mathematically by solving Eqs. 5 and 4. The intersection point is, $\left(\frac{S_0 - SS(\xi_i)}{SS'(\xi_i)} + \xi_i,\, S_0\right)$, on the arithmetic plot.

Step 6. Back-calculate the AEV through use of the relationship, $\xi = \mathrm{Log}_{10}(\psi)$. The air-entry value for the soil can be written as follows.

$$\psi_{AEV} = 10^{\frac{S_0 - SS(\xi_i)}{SS'(\xi_i)} + \xi_i} \qquad [6]$$

3 ROLE OF AEV IN THE FREDLUND, XING AND HUANG (1994) PERMEABILITY FUNCTION

Fredlund, Xing and Huang (1994) suggested a mathematical function for the estimation of the relative coefficient of permeability based on a physical model proposed by Childs and Collis-George (1950) (see Eq. 7).

$$k_r^S(\psi) = \frac{\displaystyle\int_{\ln(\psi)}^{b} \frac{S(e^y) - S(\psi)}{e^y} S'(e^y)\,dy}{\displaystyle\int_{\ln(\psi_{aev})}^{b} \frac{S(e^y) - S(\psi_{aev})}{e^y} S'(e^y)\,dy} \qquad [7]$$

where,

$k_r^s(\psi)$ = relative coefficient of permeability at soil suction of ψ. The superscript *s* means that the degree of saturation-SWCC is used for the estimation of the relative permeability in Eq. 7;

b = upper limit of integration [i.e., ln(1000000)];

y = dummy variable of integration representing the logarithm of suction;

S = degree of saturation-SWCC equation;

S' = derivative of the degree of saturation-SWCC equation;

e^y = natural number raised to the dummy variable power.

The denominator of Eq. 7 is an integral, the lower limit of the integration of which is the air-entry value, ψ_{aev}. Although the original theory (Fredlund et al. 1994) specified the air-entry value, ψ_{aev}, as the lower limit of integration, other values between a value close to zero and ψ_{aev} have been used as the starting point for integration while estimating the relative permeability function. The arbitrarily selected small value for the starting point of integration appears to have been used because no closed-form analytical procedure had been proposed for the calculation of the AEV. Details on how the integration using Fredlund et al. (1994) permeability is to be carried out can be found in the original paper.

If a suction value ψ_i between (near) zero and ψ_{aev} is used as the lower limit of integration, the permeability function of Eq. 7 takes on the form shown in Eq. 8.

$$k_{ri}^S(\psi) = \frac{\displaystyle\int_{\ln(\psi)}^{b} \frac{S(e^y) - S(\psi)}{e^y} S'(e^y)\,dy}{\displaystyle\int_{\ln(\psi_i)}^{b} \frac{S(e^y) - S(\psi_i)}{e^y} S'(e^y)\,dy} \qquad [8]$$

where, $k_{ri}^s(\psi)$ = relative coefficient of permeability at soil suction of ψ, when a suction value ψ_i is used as the lower limit of integration for the integral in the denominator of the Eq. 8.

Childs and Collis-George (1950) proposed a statistical model for estimating the coefficient of permeability based on a random variation in pore sizes. There are three common assumptions for a methodology characterizing the statistical models: (a) The porous medium may be regarded as a set of interconnected pores randomly distributed in the sample. The pores are characterized by their length scale called “the pore radius”. (b) The Hagen-Poiseuille equation is assumed valid at the level of the single pore and thus used to estimate the hydraulic conductivity of the elementary pore unit. The total hydraulic conductivity has to be determined by integration over the contributions of the filled pores. (c) The soil-water characteristic curve is considered analogous to the pore radius distribution function. The capillary law is used to uniquely relate the pore radius to the capillary head (Mualem 1986). The air-entry value of the soil corresponds to the largest pore radius. This is the theoretical reason why the air-entry value has to be used as the lower limit of integration when estimating the relative permeability using Fredlund, Xing and Huang (1994) permeability function. The change of the lower limit of integration implies a change in the largest pore radius of the soil and thus the change in the pore radius distribution function.

The relative coefficient of permeability obtained using Eq. 7 is theoretically correct and is used as the reference value. An error in the estimation of the relative permeability is introduced when using Eq. 8 along with a variety of the lower limits of integration in the denominator. The slope on the soil-water characteristic curve, (SWCC), prior to the AEV (as defined by the degree of saturation-SWCC), contributes to the error in the computed permeability function.

Figure 5 presents the relative coefficient of permeability for GE3 from Brooks and Corey (1964). The SWCC for GE3 is shown in Figure 1. The best-fitting parameters of the Fredlund and Xing (1994) function for the *S*-SWCC of GE3 are a_f = 8.20 kPa, n_f = 9.15, m_f = 0.45, ψ_r = 40 kPa. GE3 has an AEV of 7 kPa. Figure 6 presents the relative coefficient of permeability for Regina clay from Fredlund (1964). The SWCCs for Regina clay are shown in Figure 2. It is the *S*-SWCC that is used for the estimation of the relative coefficient of permeability function. The best-fitting parameters of the Fredlund and Xing (1994) function for the *S*-SWCC of Regina clay are a_f = 7105 kPa, n_f = 1.348, m_f = 0.461, ψ_r = 47238 kPa. Regina clay has an AEV of 3500 kPa.

Figure 5 illustrates a situation where the effect of the starting point for integration is small. Starting integration at any point from 0.1 kPa to the AEV results in the computation of essentially the same relative permeability function. Figure 6, on the other hand, shows how the starting point for integration can have a significant effect on the computed permeability function. The result computed by starting the integration from 0.1 kPa underestimates the permeability function by about one order of magnitude. The effect of the lower limit of integration on the calculation of the permeability function is studied in detail by the recently published paper of Zhang and Fredlund (2015).

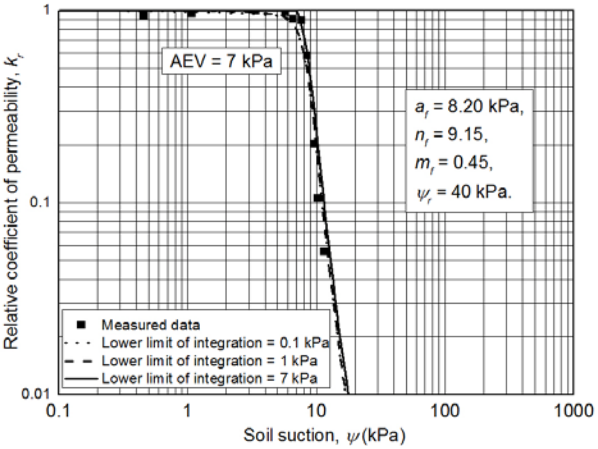

Figure 5. Relative permeability for GE3 obtained using Eq. 8 with different lower limits of integration. (Data from Brooks and Corey 1964)

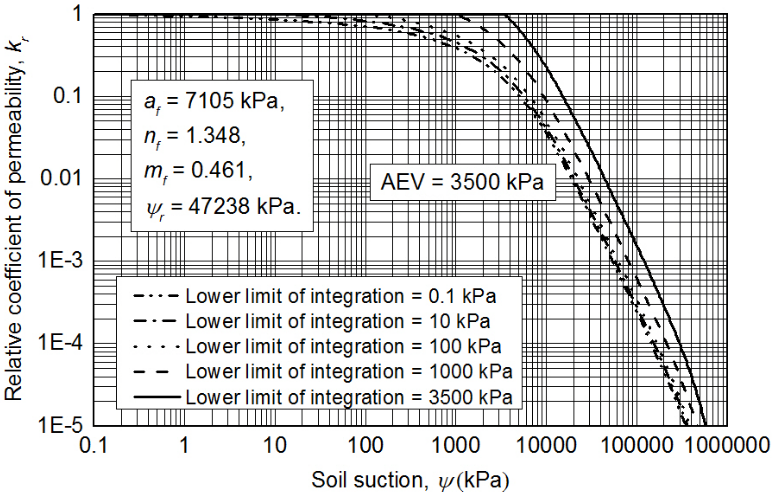

Figure 6. Relative permeability for Regina clay obtained using Eq. 8 with different lower limit of integration. (Data from Fredlund 1964)

4 THE CHOICE OF SWCC FOR THE ESTIMATION OF PERMEABILITY FUNCTION

The Fredlund, Xing and Huang (1994) permeability function was developed based on the interpretation of the SWCC. The choice of SWCC made could greatly affect the estimation results particularly when a soil's different forms of SWCC are distinct in shape and characteristics from each other. There are four forms of SWCCs, namely, gravimetric water content SWCC (*w*-SWCC), volumetric water content (θ-SWCC), instantaneous volumetric water content SWCC (θ_i-SWCC), and degree of saturation SWCC (*S*-SWCC). These four forms of SWCCs are essentially the same for no volume change soils (e.g., sands and silts) when plotted in terms of dimensionless water content versus soil suction as shown in Figure 1. For soils that change volume when suction is increased, *w*-SWCC, θ_i-SWCC, and *S*-SWCC are distinctly different from each other. θ-SWCC is of no significance for soils that change volume. Regina clay is a typical soil that undergoes volume change when soil suction is increased.

Different forms of SWCC for Regina clay are presented in Figure 2. The best-fitting parameters of Fredlund and Xing (1994) SWCC function as well as the break point or AEV for each curve shown in Figure 2 are listed in Table 1. The results presented in Figure 2 and Table 1 reveal the significant difference among *w*-SWCC, θ_i-SWCC, and *S*-SWCC for Regina clay. θ-SWCC overlaps with *S*-SWCC, but *θ*-SWCC is of no significance in the case where there is volume change. *θ*-SWCC will be omitted in the following discussion. AEV is the soil suction that features the beginning of desaturation, thus it is reasonable that AEV should be obtained from *S*-SWCC. The break points on *w*-SWCC and θ_i-SWCC are 4.4 kPa, and 40 kPa respectively, both of which are significantly smaller than the AEV of 3500 kPa on *S*-SWCC. Wrong choice of SWCC could significantly underestimate the AEV for the large volume change Regina clay.

The choice of SWCC also significantly influences the estimation results of the relative permeability function for soils that undergo volume change as soil suction is increased. Figure 7 shows a typical example of Regina clay in which the relative permeability functions estimated from different forms of SWCC present distinct differences. Results obtained from *w*-SWCC and θ_i-SWCC greatly underestimate the relative permeability compared to the estimation results obtained from *S*-SWCC. Results obtained from *w*-SWCC underestimate the relative permeability by about 6 orders of magnitude. Results obtained from θ_i-SWCC underestimate the permeability by about 3 orders of magnitude. These significant differences among the estimated permeability function are mainly resulted from the significantly different break points on each form of SWCC used as the lower limit of integration when estimating the relative permeability function by the integral formula. It is appropriate to use the *S*-SWCC for the reasonable estimation of the relative permeability function, because the degree of saturation is the factor that affects the tortuosity of the flow path through the unsaturated soil thus influencing the relative permeability of the soil (Fredlund and Zhang 2013, Zhang and Fredlund 2014).

Table 1. Break/AEV and best-fitting parameters for each form of SWCC for Regina clay (Data from Fredlund 1964)

	w-SWCC	*θ*-SWCC	θ_i-SWCC	*S*-SWCC
Initial water content, (%) ($w_s/\theta_0/S_0$)	86.1	67.12	67.12	92.57
a_f (kPa)	17.2	17.2	88.34	7105
n_f	0.871	0.871	0.6023	1.348
m_f	0.770	0.770	0.589	0.461
ψ_r (kPa)	922	922	2600	47238
Break/AEV (kPa)	4.4	4.1	40	3500

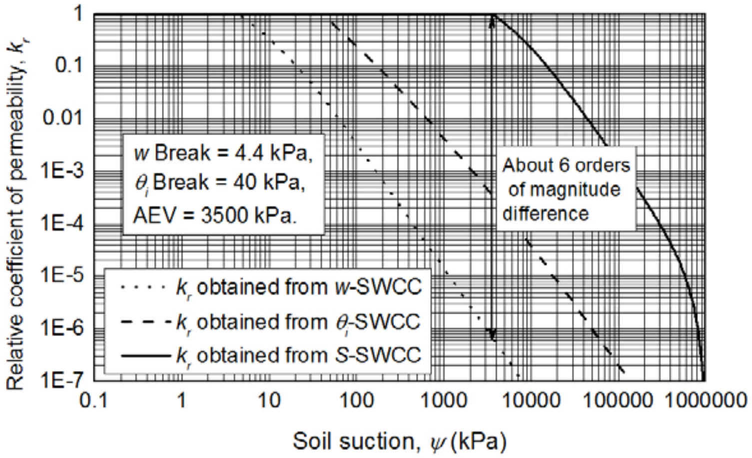

Figure 7. Relative coefficient of permeability for Regina clay estimated from different forms of SWCC using each SWCC curve's break as the lower limit of integration.

5 CONCLUSIONS

A mathematical algorithm for the calculation of the AEV for the integration purposes is proposed in this paper. The effect of the lower limit of integration on the estimation of the permeability function is studied. The results show that if a lower limit of integration used in the integral of Fredlund et al. (1994) is smaller than the AEV, the computed results will underestimate the relative coefficient of permeability. The smaller the value used for the starting point of integration compared to the AEV, the greater will be the difference between the computed results and the relative permeability. This is particularly true for the high volume change Regina clay. It is recommended that the AEV always be used as the lower limit of integration when estimating the relative permeability function with the Fredlund et al. (1994) estimation method.

The study in this paper also reveals the importance of using *S*-SWCC in the determination of AEV and the estimation of the relative coefficient of permeability especially when soils undergo volume change as soil suction changes (e.g., Regina clay).

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